B Powell
1858

AN ELEMENTARY TREATISE

ON

THE LUNAR THEORY,

WITH

A BRIEF SKETCH OF THE HISTORY OF THE PROBLEM

UP TO THE TIME OF NEWTON,

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> Et toi, sœur du Soleil, astre qui dans les cieux
> Des mortels éblouis trompais les faibles yeux,
> Newton de ta carrière a marqué les limites
> Marche, éclaire les nuits, tes bornes sont prescrites
>
> VOLTAIRE

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PREFACE.

Of all the celestial bodies whose motions have formed the subject of the investigations of astronomers, the Moon has always been regarded as that which presents the greatest difficulties, on account of the number of inequalities to which it is subject, but the frequent and important applications of the results render the Lunar Problem one of the highest interest, and we find that it has occupied the attention of the most celebrated astronomers from the earliest times.

Newton's discovery of Universal Gravitation, suggested, it is supposed, by a rough consideration of the motions of the moon, led him naturally to examine its application to a more severe explanation of her disturbances, and his Eleventh Section is the first attempt at a theoretical investigation of the Lunar inequalities The results he obtained were found to agree very nearly with those determined by observation, and afforded a remarkable confirmation of the truth of his great principle, but the geometrical methods which he had adopted seem inadequate to so complicated a theory, and recourse has been had to analysis for a complete determination of the disturbances, and for a knowledge of the true orbit

The following pages will, it is hoped, form a proper introduction to more recondite works on the subject the difficulties which a person entering upon this study is most likely to stumble at, have been dwelt upon at considerable length, and though different methods of investigation have been employed by different astronomers, the difficulties met with are nearly the same, and the principle of successive approximation is common to all In the present work, the approximation is carried to the second order of small quantities, and this, though far from giving accurate values, is amply sufficient for the elucidation of the method

The differences in the analytical solutions arise from the various ways in which the position of the moon may be indicated by altering the system of coordinates to which it is referred, or again, in the same system, by choosing different quantities for independent variables

D'Alembert and Clairaut chose for coordinates the projection of the radius vector on the plane of the ecliptic and the longitude of this projection To form the differential equations, the true longitude was taken for independent variable

To determine the latitude, they, by analogy to Newton's method, employed the differential variations of the motion of the node and of the inclination of the orbit.

Laplace, Damoiseau, Plana, and also Herschel and Airy in their more elementary works, have found it more convenient to express the variations of the latitude directly, by an equation of the same form as that of the radius vector

Lubbock and Pontécoulant, taking the same coordinates of the moon's position, make the time the independent variable, and when it is desired to carry the approximation to a high order, this method offers the advantage of not requiring any reversion of series

Poisson proposed the method used in the planetary theory, that is, to determine the variation in the elements of the moon's orbit, and thence to conclude the corresponding variations of the radius vector, the longitude, and the latitude.

The selection of the method followed in the present work, which is the same as that of Airy, Herschel, &c, was made on account of its simplicity moreover, it is the method which has obtained in this university, and it is hoped that it may prove of service to the student in his reading for the examination for Honours In furtherance of this object, one of the chapters (the sixth) contains the physical interpretation of the various important terms in the radius vector, latitude, and longitude [*]

The seventh chapter, or Appendix, contains some of the most interesting results in the terms of the higher orders, among which will be found the values of c and g completely obtained to the third order

The last chapter is a brief historical sketch of the Lunar Problem up to the time of Newton, containing an account of the discoveries of the several inequalities and of the methods by which they were represented, those only being mentioned which, as the theory has since verified, were real onward steps. The perusal of this chapter will shew to what extent we are indebted to our great philosopher; at the same time we cannot fail being impressed with reverence for the genius and perseverance of the men who preceded him, and whose elaborate and multiplied hypotheses were in some measure necessary to the discovery of his simple and single law.

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[*] See the Report of the "Board of Mathematical Studies" for 1850

I take this opportunity of acknowledging my obligations to several friends, whose valuable suggestions have added to the utility of the work

HUGH GODFRAY

Cambridge, April 16th, 1853

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LUNAR THEORY.

CHAPTER I.

INTRODUCTION

BEFORE proceeding to the consideration of the moon's motion, it will be desirable to say a few words on the law of attractions, and on the peculiar circumstances which enable us to simplify the present investigation

1 The law of universal gravitation, as laid down by Newton, is that "*Every particle in the universe attracts every other particle, with a force varying directly as the mass of the attracting particle and inversely as the square of the distance between them*"

The truth of this law cannot be established by abstract reasoning; but as it is found that the motions of the heavenly bodies, calculated on the assumption of its truth, agree more and more closely with the observed motions as our calculations are more strictly performed, we have every reason to consider the law as an established truth, and to attribute any slight discrepancy between the results of calculation and observation to instrumental errors, to an incomplete analysis, or to our ignorance of the existence of some of the disturbing causes.

Of the last cause of deviation there is a remarkable instance in the recent discovery of the planet Neptune, for our acquaintance with which as one of the bodies of our system,* we are indebted to the perturbations it produced in the calculated places of the planet Uranus. These perturbations were too great to be attributed wholly to errors of instruments or of calculation, and therefore, either the law of universal gravitation was here at fault, or some unknown body was disturbing the path of the planet. This last supposition, in the powerful hands of Messrs Adams and Le Verrier, led to the detection of Neptune by solving the difficult inverse problem, viz.: Given the perturbations caused by a body, determine, on the assumption of the truth of Newton's law, the orbit and position of the disturbing body.

Evidence so strong as this forces us to admit the correctness of the assumption, and we must now consider how this law, combined with the laws of motion, will enable us to investigate the circumstances of the moon's motion, and to assign her position at any time when observation has furnished the requisite data.

2. The problem in its present form would be one of extreme, if not insurmountable difficulty, if we had to take into account simultaneously the actions of the earth, sun, planets, &c. on the moon; but fortunately the earth's attraction, on account of its proximity, is much greater than the *disturbing*† force of the sun or of any planet, these disturbing forces being so small compared with the absolute force of the earth, that the squares and products of the effects they produce may be neglected, except in extreme cases; and there is a principle, called the "principle of the superposition of small motions," which shows that in such a case the disturbing forces may be considered separately, and the algebraic sum of the

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* It had been seen by Dr. Lamont at Munich, one year before its being known to be a planet. "Solar System, by J. R. Hind."

† Since the sun attracts both the earth and moon, it is clear that its effects on the moon's motion relatively to the earth or the disturbing force will not be the same as the absolute force on either body. This will be fully investigated in Arts. (9) and (24).

disturbances so obtained will be the same as the disturbance due to the simultaneous action of all the forces

Principle of Superposition of Small Motions

3 Let a particle be moving under the action of any number of forces some of which are very small, and let A (fig 1) be the position of the particle at any instant Let two of these small forces m_1, m_2 be omitted, and suppose the path of the particle under the action of the remaining forces to be AP in any given time

Let AP_1 be the path which would have been described in the same time if m_1 also had acted, AP_1 differing very slightly from AP, and PP_1 being the disturbance

Similarly, if m_2 instead of m_1 had acted, suppose AP_2 to have represented the disturbed path (AP, AP_1, AP_2 are not necessarily in the same plane, nor even plane curves), PP_2 being the disturbance

Lastly, let AQ be the actual path of the body when both m_1 and m_2 act Join P_1Q

Now, since the path AP_1 very nearly coincides with AP, the disturbance P_1Q, due to the action of m_2 on the path AP_1, can differ in magnitude and direction from the disturbance PP_2, due to the action of the same force on the path AP, only by a quantity of the first order compared with PP_2 or of the second order compared with AP, and it may therefore be neglected Therefore P_1Q is parallel and equal to PP_2.

Hence the projection of the whole disturbance PQ on any straight line, being equal to the algebraical sum of the projections of PP_1 and P_1Q, will be equal to the algebraical sum of the projections of the separate disturbances PP_1, PP_2.

Now if there are three small disturbing forces m_1, m_2, m_3, we may consider the joint action of the two m_2, m_3, as one small disturbing force; therefore, by what precedes, the total disturbance along any axis will be the sum of the separate disturbances of m_1 and of the system m_2, m_3; but this last is the sum of the separate disturbances of m_2 and m_3: therefore the whole disturbance equals the sum of the three separate disturbances

This reasoning can evidently be extended to any number of forces, and if x, y, z be the coordinates of the disturbed particle, $\phi(x, y, z)$ any function of x, y, z, the disturbance produced in $\phi(x, y, z)$ will be

$$\delta\phi(x, y, z) = \frac{d\phi}{dx}\,\delta x + \frac{d\phi}{dy}\,\delta y + \frac{d\phi}{dz}\,\delta z, \text{ omitting } (\delta x)^2, \text{ \&c },$$

where $\delta x = \delta x_1 + \delta x_2 + \quad =$ sum of disturbances along axis of x due to separate forces,

$\delta y = \delta y_1 + \delta y_2 + \quad . =$. along axis of y,

$\delta z = \delta z_1 + \delta z_2 + \quad =$. along axis of z,

therefore
$$\delta\phi(x, y, z) = \frac{d\phi}{dx}\,\delta x_1 + \frac{d\phi}{dy}\,\delta y_1 + \frac{d\phi}{dz}\,dz_1$$
$$+ \frac{d\phi}{dx}\,\delta x_2 + \frac{d\phi}{dy}\,\delta y_2 + \frac{d\phi}{dz}\,\delta z_2 + \text{\&c.}$$
$$= \delta_1\phi(x, y, z) + \delta_2\phi(x, y, z) + \text{\&c.},$$

or total disturbance equals sum of separate disturbances, which proves the proposition.

4 Since $\phi(x, y, z)$ may be the radius vector, or the latitude or longitude of the disturbed body, it follows that the total disturbance in any of these elements is the sum of the partial disturbances.

Therefore in determining the motion of a secondary relative to its primary, as in the present case of the moon about the earth, where the disturbing effects produced by the sun and planets are small, we may consider them one at a time, and hence the famous problem of the *Three Bodies.*

The planets being small and distant, their effect on the motion of the moon will not be of sufficient intensity to affect the order of approximation to which it is intended to carry the solution in the following pages, and our problem is reduced to the consideration of the three bodies, the sun, earth, and moon, acting on one another according to the law of universal gravitation

5. But we must still prove another proposition, without which the problem would scarcely, though reduced to three bodies, be less complicated than in its most general form.

Newton's law refers to *particles*, whereas the sun, earth, and moon are *large* spherical bodies, and it becomes necessary to examine the mutual action of such bodies Now, it happens that with this law of force, the attraction of one sphere on another can be correctly obtained, and leaves the question in exactly the same state as if they were particles (*Princip* lib. I. prop. 75)

Attractions of Spherical Bodies

6. Let P (fig. 2) be a particle situated at a distance $OP = a$ from the centre of a *uniform* attracting sphere whose density is ρ and radius $OA = c$ $a > c$, the particle being without the sphere.

Let the whole sphere be divided into circular laminae by planes perpendicular to OP. Let SQ be one of these $PS = x$, $PQ = z$, and thickness of lamina $= \delta x$

Next, let this lamina be divided into concentric rings. Let $RS = r$ be the radius of one of these rings and δr its breadth, $\angle RPO = \theta$;

therefore
$$r = x \tan\theta,$$
$$\delta r = x \sec^2\theta . \delta\theta$$

The attraction of an element R of this ring on the particle P will be $\dfrac{\text{mass of element}}{PR^2}$ along PR, and the resolved part of this along PO will be $\dfrac{\text{mass of element}}{x^2 \sec^2\theta} \cos\theta$.

But the resultant attraction of the whole ring will clearly be the sum of the resolved parts along PO of the attractions of its constituent elements; therefore,

$$\text{attraction of ring} = \frac{2\pi\rho r\ \delta r\ \delta x}{x^2 \sec^2\theta} \cos\theta = 2\pi\rho \sin\theta\, \delta x . \delta\theta\,;$$

therefore, attraction of whole lamina $SQ = 2\pi\rho\delta x \int_0^{\cos^{-1}\frac{x}{z}} \sin\theta \, d\theta$

$$= 2\pi\rho\delta x \left(1 - \frac{x}{z}\right)$$

Again, $z^2 = x^2 + c^2 - (a-x)^2 = 2ax - (a^2 - c^2)$;

therefore $z\delta z = a\delta x$,

and attraction of lamina $= 2\pi\rho \left(\frac{z\delta z}{a} - \frac{z^2 + a^2 - c^2}{2a^2} \delta z\right)$,

attraction of whole sphere $= 2\pi\rho \left\{\frac{z^2}{2a} - \frac{z^3}{6a^2} - \frac{(a^2 - c^2) z}{2a^2}\right\}$

(from $z = a - c$ to $z = a + c$)

$$= \frac{4\pi\rho c^3}{3a^2}$$

$$= \frac{M}{a^2},$$

where $M =$ mass of sphere $= \frac{4\pi\rho c^3}{3}$

Hence, *the attraction of the whole sphere is precisely the same as if the whole mass were condensed into its centre*

COR 1 The attraction of a shell radius $= c$ and thickness δc will be obtained from the preceding expression by differentiating it with respect to c, and is

$$\text{attraction of shell} = \frac{4\pi\rho c^2 \delta c}{a^2} = \frac{\text{mass of shell}}{a^2}$$

COR 2 Therefore, *the attraction of a* heterogeneous *sphere on an external particle will be the same as if the whole mass were condensed into its centre, provided the density be the same at all points equally distant from the centre*, for then the whole sphere may be considered as the aggregate of an infinite number of uniform shells, and by Cor. 1, each acts as if condensed into its centre

7. Let us now consider the case of one sphere attracting another Suppose P in the preceding article to be an elementary particle of a sphere M', whose centre O' suppose at a distance a from O. Then, since action and reaction are equal and opposite, P will attract the whole sphere M just as it would do a particle of mass M placed at O; the same is true of all the elementary particles which compose the sphere M', therefore the sphere M' will attract the sphere M as if the whole mass of the latter were condensed into its centre O but the attraction of the sphere M' on a particle O is the same as if the attracting sphere were condensed into its centre O'; therefore,

Two spheres attract one another as if the whole matter of each sphere were collected at its centre.

8 This remarkable result, which, as may be shewn, holds only when the law of attraction is that of the inverse square of the distance, or that of the direct distance, or a combination of these by addition or subtraction, reduces the problem of the sun, earth, and moon to that of three particles;—the slight error due to the bodies not being perfect spheres will here be neglected, being of an order higher than that to which we intend to carry the present investigation: this error however, though very small, is appreciable, and if a nearer approximation were required, it would be necessary to have regard to this circumstance (See Appendix, Art 100)

CHAPTER II.

MOTION RELATIVE TO THE EARTH

9 When a number of particles are in motion under their mutual attractions or other forces, and the motion relatively to one of them is required, we must bring that one to rest, and then keep it at rest without altering the relative motions of the others with respect to it

Now, firstly, the chosen particle will be brought to rest by giving it at any instant a velocity equal and opposite to that which it has at that instant, secondly, it will be kept at rest by applying to it accelerating forces equal and opposite to those which act upon it

Therefore give the same velocity and apply the same accelerating forces to all the bodies of the system, and the absolute motions about the chosen body, which is now at rest, will be the same as their relative motions previously.

Problem of Two Bodies

10. As the sun disturbs the moon's motion with respect to the earth, it is important to know what that motion would have been if no disturbance had existed, or generally .—

Two bodies attracting one another with forces varying directly as the mass and inversely as the square of the distance, to determine the orbit of one relatively to the other.

Let M, M' be the masses of the bodies, r the distance between them at any time t, M' being the body whose motion relatively to M is required.

The accelerating force of M on M' equals $\frac{M}{r^2}$ acting towards

M, while that of M' on M equals $\frac{M'}{r^2}$ in the opposite direction Therefore, by the principle above stated, we must apply to both M and M' accelerating forces equal and opposite to this latter force, and M' will move about M fixed, the accelerating force on M' being $\frac{M+M'}{r^2} = \mu u^2$, if $\mu = M + M'$ and $r = \frac{1}{u}$

Hence,
$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} = \frac{\mu}{h^2},$$
where $h = r^2 \frac{d\theta}{dt}$ = twice the area described in a unit of time,
$$u = \frac{\mu}{h^2}\{1 + e\cos(\theta - \alpha)\},$$
e and α being constants to be determined by the circumstances of the motion at any given time

This is the equation to a conic section referred to its focus, the eccentricity being e, the semi-latus rectum $\frac{h^2}{\mu}$, and the angle made by the apse line with the prime radius α

In the relative motion of the moon, or in that of the sun about the earth, the orbit would, as observation informs us, be an ellipse with small eccentricity, that of the moon being about $\frac{1}{20}$ and that of the sun $\frac{1}{60}$

11. The angle $\theta - \alpha$ between the radius vector and the apse line is called the *true anomaly*

If n is the angular velocity of a radius vector which moving uniformly would accomplish its revolution in the same time as the true one, both passing through the apse at the same instant, then $nt + \varepsilon - \alpha$ is called the *mean anomaly*, ε being a constant depending on the instant when the body is at the apse, its value being also equal to the angle between the prime radius and the uniformly revolving one when $t = 0$.

Thus, if $M\Upsilon$ (fig 3) be the fixed line or prime radius,

A the apse,

M' the moving body at time t,

$M\mu$ the uniformly revolving radius at same time, the direction of motion being represented by the arrow

Let MD be the position of $M\mu$ when $t = 0$,

then $\quad \Upsilon MD = \varepsilon$ and is called the epoch,*

$$DM\mu = nt,$$

$$\Upsilon MA = \alpha = \text{longitude of the apse};$$

therefore, mean anomaly $= AM\mu = nt + \varepsilon - \alpha$,

true anomaly $= AMM' = \Upsilon MM' - \Upsilon MA = \theta - \alpha$

12. *To express the* mean *anomaly in terms of the* true *in a series ascending according to the powers of* e, *as far as* e^2

$$n = \frac{2\pi}{\text{periodic time}} = 2\pi - \frac{2\text{ area}}{h}$$

$$= \frac{2\pi h}{2\pi ab} = \frac{h}{a^2\sqrt{(1-e^2)}},$$

therefore $$h = na^2(1-e^2)^{\frac{1}{2}},$$

$$\frac{dt}{d\theta} = \frac{r^2}{h} = \frac{a^2(1-e^2)^2}{h} \; \frac{1}{\{1 + e\cos(\theta-\alpha)\}^2}$$

$$= \frac{1}{n}(1-e^2)^{\frac{3}{2}}\{1+e\cos(\theta-\alpha)\}^{-2}$$

$$= \frac{1}{n}(1 - \tfrac{3}{2}e^2)\{1 - 2e\cos(\theta-\alpha) + 3e^2\cos^2(\theta-\alpha)\}$$

$$= \frac{1}{n}\{1 - 2e\cos(\theta-\alpha) + \tfrac{3}{2}e^2\cos 2(\theta-\alpha)\};$$

therefore $\quad nt + \varepsilon = \theta - 2e\sin(\theta-\alpha) + \tfrac{3}{4}e^2\sin 2(\theta-\alpha)$,

or $\quad (nt+\varepsilon-\alpha) = (\theta-\alpha) - 2e\sin(\theta-\alpha) + \tfrac{3}{4}e^2\sin 2(\theta-\alpha)$,

the required relation.

13. *To express the* true *anomaly in terms of the* mean *to the same order of approximation*

$$\theta - \alpha = nt + \varepsilon - \alpha + 2e\sin(\theta-\alpha) - \tfrac{3}{4}e^2\sin 2(\theta-\alpha) \quad .\,(1);$$

$\therefore \quad \theta - \alpha = nt + \varepsilon - \alpha \quad$ first approximation

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* The introduction of the epoch is avoided in the Lunar theory by a particular assumption (Art 34), but in the Planetary it forms one of the elements of the orbit

Substituting this in the first small terms of (1), we get

$$\theta - \alpha = nt + \varepsilon - \alpha + 2e \sin(nt + \varepsilon - \alpha) \quad \text{a second approximation.}$$

Substitute the second approximation in that small term of (1) which is multiplied by e, and the first approximation in that multiplied by e^2, the result will be correct to that term, and gives

$$\begin{aligned}\theta - \alpha &= nt + \varepsilon - \alpha + 2e \sin\{nt + \varepsilon - \alpha + 2e \sin(nt + \varepsilon - \alpha)\} \\ &\qquad - \tfrac{3}{4}e^2 \sin 2(nt + \varepsilon - \alpha) \\ &= nt + \varepsilon - \alpha + 2e \sin(nt + \varepsilon - \alpha) \\ &\qquad + 4e^2 \cos(nt + \varepsilon - \alpha) \sin(nt + \varepsilon - \alpha) - \tfrac{3}{4}e^2 \sin 2(nt + \varepsilon - \alpha) \\ &= nt + \varepsilon - \alpha + 2e \sin(nt + \varepsilon - \alpha) + \tfrac{5}{4}e^2 \sin 2(nt + \varepsilon - \alpha),\end{aligned}$$

the required relation.

The development could be carried on by the same process to any power of e, but in what follows we shall not require it further than e^2

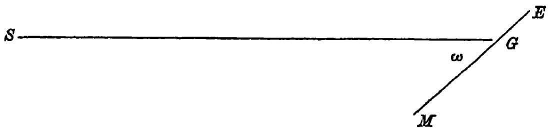
Problem of Three Bodies

14. In order to fix the position of the moon with respect to the centre of the earth, which, by means of the process described in Art (9), is reduced to and kept at rest, we must have some determinate invariable plane passing through the earth's centre to which the motion may be referred

The plane which passes through the earth's centre and the direction of the sun's motion at any instant is called the *true ecliptic*, and as a first rough result of calculation, obtained on supposition of the sun and earth being the only bodies in the universe, this plane, in which, according to the last section, an elliptic orbit would be described, is a fixed plane but this is no longer the case when we take into account the disturbances produced by the moon and planets, and it becomes necessary to substitute some other plane of reference unaffected by these disturbances

Theory teaches us that such a plane exists,* but as its determination can only be the work of time, the following theorem will supply us with a plane whose motion is extremely slow, and it may for a long period and to a degree of approximation far beyond that to which we shall carry our investigations, be considered as fixed and coinciding with its position at present

15 *The centre of gravity of the earth and moon describes relatively to the sun an orbit very nearly in one plane and elliptic,* the square of the ratio of the distances of the moon and sun from the earth being neglected †



Let S, E, M be the centres of the sun, earth, and moon, G the centre of gravity of the two last Now the motion of G is the same as if the whole mass $E+M$ were collected there and acted on by forces equal and parallel to the moving forces which act on E and M The whole force on G is therefore in the plane SEM, join SG

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* See *Poinsot*, "*Theorie et determination de l'equateur du système solaire,*" where he proves that an invariable plane exists for the solar system, that is, a plane whose position relatively to the fixed stars will always be the same whatever changes the orbits of the planets may experience, but as its position depends on the moments of inertia of the sun, planets, and satellites, and therefore on their internal conformation, it cannot be determined *à priori*, and ages must elapse before observation can furnish sufficient data for doing so *à posteriori*

This result Poinsot obtains on the supposition that the solar system is a free system, but it is possible, as he furthermore remarks, nay probable, that the stars exert some action upon it, it follows that this *invariable* plane may itself be variable, though the change must, according to our ideas of time and space, be indefinitely slow and small.

† This ratio is about $\frac{1}{400}$ and, as we shall see Art (21), such a quantity we shall consider as of the 2nd order of small quantities, and its square therefore of the 4th order Our investigations are carried to the 2nd order only

Let $\angle SGM = \omega$, and let m' be the sun's absolute force

$$\text{Moving force on } E = \frac{m'\,E}{SE^2} \text{ in } ES,$$

$$\text{moving force on } M = \frac{m'\,M}{SM^2} \text{ in } MS$$

These* applied to G parallel to themselves are equivalent to

$$\frac{m'\,E\,GE}{SE^3} - \frac{m'\,M\,GM}{SM^3} \text{ in direction } GM,$$

and $$\frac{m'\,E\,SG}{SE^3} + \frac{m'\,M\,SG}{SM^3} \quad . \qquad GS,$$

But $$SE^2 = SG^2 + GE^2 + 2\,SG\,GE\cos\omega,$$
$$SM^2 = SG^2 + GM^2 - 2\,SG\,GM\cos\omega$$

Hence $$\left.\begin{aligned} \frac{1}{SE^3} &= \frac{1}{SG^3} - \frac{3\,GE}{SG^4}\cos\omega \\ \frac{1}{SM^3} &= \frac{1}{SG^3} + \frac{3\,GM}{SG^4}\cos\omega \end{aligned}\right\} \text{omitting } \left(\frac{EM}{SG}\right)^2.$$

Now $E\,GE = M.GM = (M+E)\,\dfrac{GM\,GE}{ME}$. Therefore the accelerating force in the direction GM

$$= -\frac{3m'\,GM\,GE}{SG^4}\cos\omega$$

$$= -3\,(\text{accelerating force of sun on } G)\,\frac{GM}{SG}\cdot\frac{GE}{SG}\cos\omega$$

$= 0$ according to the standard of approximation adopted.

Hence the only force on G is a central force tending to S, and therefore the motion of G will be in one plane

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[*] In strictness it would be necessary, since we have brought S to rest, to apply to both M and E, and therefore to G, accelerating forces equal and opposite to those which E and M themselves exert on S, but the mass of S is so large compared with those of E and M, that we may safely neglect these forces in this approximate determination of the path of G, the error being of a still higher order than that introduced by the neglect of $\left(\frac{EM}{SG}\right)$

Again, the accelerating force in GS

$$= \frac{m'SG}{M+E}\left\{\frac{E+M}{SG^3} - \frac{3(E\,GE - M\,GM)}{SG^4}\cos\omega\right\}$$

$$= \frac{m'}{SG^2} \text{ to the same approximation.}$$

Therefore the orbit of G about S is very approximately an ellipse with S in the focus, and the plane of this ellipse is, as far as our investigations are concerned, a fixed plane when S is fixed

This fixed plane is called the *plane of the ecliptic*, or simply the *ecliptic*

16 A plane through the earth's centre parallel to the ecliptic will be the plane of reference we require (14) and will become a fixed plane when we bring the earth's centre to rest, the ecliptic then making small monthly oscillations from one side to the other of our fixed plane

17. The sun will have a latitude always of the same name as that of the moon, and deducible from it, when ES, EM, and the ratio of the masses of the earth and moon are known. For if $S'EM'$ be this fixed plane through E (fig. 4), S', G', M', the projections of S, G, M,

then, sin(sun's lat)

$$= \sin SES' = \frac{SS'}{ES} = \frac{GG'}{ES} = \frac{EG\sin(\text{moon's lat })}{ES}$$

$$= \frac{M}{E+M}\,\frac{EM}{ES}\sin(\text{moon's lat })$$

Now, from observation it is known that M is about $\frac{1}{80}$ of E,

and EM $\frac{1}{400}$ of ES,

$$\text{therefore } \sin(\text{sun's lat.}) = \frac{\sin(\text{moon's lat })}{32400} \text{ nearly}$$

And as the moon's latitude never exceeds 5° 9′, the sun's latitude will always be less than 1″

Again, with respect to the sun's longitude let $E\Upsilon$ be the direction of the first point of Aries,—that is, a fixed line in our plane of reference from which the longitudes of the bodies are reckoned $\Upsilon ES' = \theta'$ the sun's longitude

The difference in the sun's longitude, as seen from E or from G, will be the angle $ES'G'$.

$$\sin ES'G' = \frac{EG'}{ES'} \sin\omega = \frac{\sin\omega}{32400} \text{ approximately, if } EG'S' = \omega;$$

$$\text{therefore } \sin ES'G' \text{ never exceeds } \tfrac{1}{32400},$$

$$\text{therefore } ES'G' \text{ is a small angle not exceeding } 7''$$

$$\text{Also } ES' - S'G' < EG' < \tfrac{1}{32400} S'G'.$$

Now, by assuming the longitude and distance of the sun as seen from E to be the same as when seen from G, we commit the above small errors in the position of S, that is, we assume the sun to be at S'' instead of S, $S'S''$ being drawn equal and parallel to $G'E$ If our object were the determination of the sun's position, it would be necessary to take this into account, but the consequent small errors introduced in the disturbance of the moon will clearly, on account of the great distance of the sun, be of a far higher order, and may therefore be neglected.

18. Hence we may assume that the motion of the sun about the earth at rest is an ellipse having the earth for its focus, and its equation

$$u' = a'\{1 + e' \cos(\theta' - \zeta)\},$$

and we are safe that no appreciable error will ensue in the determination of the moon's place *

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* That is, as far as the three bodies alone are concerned,—but, since the attractions of the planets may, and in fact do, disturb the elliptic orbit of the sun about G, the same cause will disturb the *assumed* orbit about E. A remarkable result of this disturbance is noticed in Appendix, Art (99)

CHAPTER III

RIGOROUS DIFFERENTIAL EQUATIONS OF THE MOON'S MOTION AND APPROXIMATE EXPRESSIONS OF THE FORCES

19 The earth having been reduced to rest by the process described in Art (9), its centre may be taken as origin of coordinates, the fixed plane of reference as plane of xy and the line $E\Upsilon$ as axis of x (fig 5)

Let r, θ be the coordinates of the projection M' of the moon on the plane xy, s the *tangent* of the moon's latitude MEM'. Also let the accelerating forces which act on the moon be resolved into these three.

P parallel to the projected radius $M'E$ and *towards* the earth,

T parallel to the plane xy, perpendicular to P and in the direction of θ *increasing*,

S perpendicular to the fixed plane and *towards* it

$$\frac{d^2x}{dt^2} = -P\frac{x}{r} - T\frac{y}{r},$$

$$\frac{d^2y}{dt^2} = -P\frac{y}{r} + T\frac{x}{r};$$

whence

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = Tr,$$

$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} = -Pr,$$

also

$$\frac{d^2z}{dt^2} = -S.$$

And introducing polar coordinates, these equations become

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = Tr \quad . \quad . \quad . \quad . \quad . \quad (\text{i}),$$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P. \quad \ldots\ldots \quad (\text{ii}),$$

$$\frac{d^2(rs)}{dt^2} = -S\ldots\ldots\ldots\ldots\ldots \quad (\text{iii})$$

20 These three equations for determining the moon's motion take the time t for independent variable, but it will be more convenient in the following process to consider the longitude as such, and our next step will be to change the independent variable from t to θ.

From (i) we get

$$r^2\frac{d\theta}{dt}\,\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = Tr^3\frac{d\theta}{dt};$$

therefore $$\left(r^2\frac{d\theta}{dt}\right)^2 = h^2 + 2\int Tr^3 d\theta,$$

$$= H^2 \text{ suppose, whence } H\frac{dH}{d\theta} = Tr^3$$

Therefore $$\frac{d\theta}{dt} = \frac{H}{r^2} = Hu^2, \quad \text{if } u = \frac{1}{r},$$

$$\frac{dt}{d\theta} = \frac{1}{Hu^2} = \frac{1}{hu^2\sqrt{\left(1+2\int\frac{T}{h^2u^3}\,d\theta\right)}} \ldots\ldots\ldots (\alpha).$$

Again, $$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = -H\frac{du}{d\theta},$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-H\frac{du}{d\theta}\right) = -Hu^2\frac{d}{d\theta}\left(H\frac{du}{d\theta}\right).$$

Substitute in (ii),

therefore $$Hu^2\frac{d}{d\theta}\left(H\frac{du}{d\theta}\right) + H^2u^3 = P,$$

or $$H^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) + u^2\frac{du}{d\theta}H\frac{dH}{d\theta} = P \ldots\ldots\ldots\ldots (\text{A});$$

and writing for H^2 and for $H\frac{dH}{d\theta}$ their values $h^2+2\int\frac{T}{u^3}\,d\theta$ and $\frac{T}{u^3}$ the equation becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{\frac{P}{h^2u^2}-\frac{T}{h^2u^3}\frac{du}{d\theta}}{1+2\int\frac{T}{h^2u^3}\,d\theta}\quad\ldots\ldots\quad\ldots\ldots\quad(\beta)$$

This is called the *differential equation of the moon's radius vector*

Lastly,

$$-S=\frac{d^2\left(\frac{s}{u}\right)}{dt^2}=\frac{d}{dt}\left\{\frac{u\frac{ds}{d\theta}-s\frac{du}{d\theta}}{u^2}\cdot\frac{d\theta}{dt}\right\}=\frac{d}{dt}\left\{H\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\right\}$$

$$=H^2u^2\left(u\frac{d^2s}{d\theta^2}-s\frac{d^2u}{d\theta^2}\right)+H\frac{dH}{d\theta}u^2\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right),$$

from (A), $Ps=H^2u^2\left(s\frac{d^2u}{d\theta^2}+us\right)+H\frac{dH}{d\theta}u^2s\frac{du}{d\theta}$;

therefore $Ps-S=H^2u^3\left(\frac{d^2s}{d\theta^2}+s\right)+H\frac{dH}{d\theta}u^3\frac{ds}{d\theta}$;

therefore

$$\frac{d^2s}{d\theta^2}+s=\frac{\frac{Ps-S}{h^2u^3}-\frac{T}{h^2u^3}\frac{ds}{d\theta}}{1+2\int\frac{T}{h^2u^3}\,d\theta}\quad\ldots\ldots\ldots\quad(\gamma)$$

This is the *differential equation of the moon's latitude*

If the three equations (α), (β), (γ) could be integrated under these general forms, then, since they are perfectly rigorous, the problem of the moon's motion would be completely solved; for as only four variables u, θ, s, and t are involved (the accelerating forces P, T, and S are functions of these), the values of three of them, as u, θ, s, could be obtained in terms of the fourth t; that is, the radius vector, longitude, and latitude would be known corresponding to a given time.

21. But the integration has never yet been effected, except for particular values of P, T, and S; and the method which we are in consequence forced to adopt, is that of successive approximation, by which the values of u, θ, and s are obtained

in a series, the terms proceeding according to ascending powers of small fractions, some one being chosen as a standard with which all others are compared, and the order of the approximation is estimated by the highest power of the small fractions retained.

It is usual to consider $\frac{1}{20}$ as a small fraction of the first order,
consequently $\frac{1}{20}$ of $\frac{1}{20} = \frac{1}{400}$ is . . . . second
$\frac{1}{8000}$ is third .
and so on, other fractions being considered as of the 1st, 2nd, &c orders, according as they more nearly coincide with $\frac{1}{20}$, $\frac{1}{400}$, &c

22 It is necessary therefore, before we can approximate at all, that we should have a previous knowledge (a rough one is sufficient) of the values of some of the quantities involved in our investigations; and for this knowledge we must have recourse to observation.

We shall therefore assume as data the following results of observation:

(1) The moon moves in longitude about thirteen times as fast as the sun The accurate ratio of the *mean* motions in longitude represented by m is therefore about $\frac{1}{13}$, and may be considered as of the 1st order *

(2) The sun's distance from the earth is about 400 times as great as the moon's distance.

Hence the ratio of the mean distances $= \frac{1}{400}$ is of the second order. †

(3) The eccentricity e' of the elliptic orbit which the sun approximately describes about the earth is about $\frac{1}{60}$, and this, approaching nearer in value to $\frac{1}{20}$ than to $\frac{1}{400}$, will be considered as of the 1st order

---

* This approximate value of m is easily obtained,—the moon is found to perform the tour of the heavens, returning to the some position among the fixed stars, in about 27⅓ days, the sun takes 365¼ days to accomplish the same journey

† The distances of the luminaries may be calculated from their horizontal parallaxes, found by observations made at remote geographical stations

(4) During one revolution, the moon moves pretty accurately in a plane inclined to the plane of the ecliptic at an angle whose tangent is about $\frac{1}{12}$, and therefore of the 1st order *

(5) Its orbit in this plane is very nearly an ellipse having the centre of the earth in its focus, and whose eccentricity is about equal to our standard of small fractions of the 1st order, viz. $\frac{1}{20}$; and this will also be very nearly true of the projection of the orbit on the plane of the ecliptic.†

To calculate the values of P, T, S

23 We are now in possession of the data requisite for beginning our approximations, and we shall proceed to the determination of the values of P, T, and S in terms of the coordinates of the positions of the sun and moon.

Let S, E, M (fig. 6) be the centres of the sun, earth, and moon,

m', E, M their masses,

E', M', the projections on the plane of the ecliptic,

G the centre of gravity of E and M,

$\mu = E + M$,

$MGM' = \tan^{-1}s =$ moon's latitude,

$E'\Upsilon$ the direction of the first point of Aries,

$SG = r' = \frac{1}{u'}$, $\angle\, \Upsilon E'S = \theta' =$ longitude of sun,

---

* That the moon's orbit during one revolution is very nearly a plane inclined as we have stated, will be found by noting its position day after day among the fixed stars, and the rules of Spherical Trigonometry will easily enable us to verify both facts, the sun's path having previously been ascertained in a similar way

† The elliptic nature and value of the eccentricity of the moon's orbit may be found by daily observation of her parallax, whence her distance from the earth's centre may be determined corresponding observations of her place in the heavens being taken, and corrected for parallax to reduce them to the earth's centre, will determine her angular motion Lines proportional to the distances being then drawn from a point in the proper directions, the extremities mark out the form of the moon's orbit

The same method applies to the determination of the eccentricity of the sun's orbit

$$M'E' = r = \frac{1}{u};\ \angle \Upsilon E'M' = \theta = \text{longitude of moon},$$

$$\therefore\ SE'M' = \theta - \theta' = \text{difference of longitude of sun and moon.}$$

The forces we have to take into account are, according to Art (9), the forces which act directly on M, and forces equal and opposite to those which act on E,—these last being applied to the whole system so that E may be a fixed point

Attraction of S upon $M = \dfrac{m'}{SM^2}$ in MS, equivalent to

$$\left\{\begin{array}{ll} m'\dfrac{MG}{SM^3} \ldots\ldots\ldots\ldots\ldots & \text{in } MG, \\ m'\dfrac{r'}{SM^3} \ldots\ldots\ldots\ldots & \text{parallel to } GS; \end{array}\right.$$

attraction of E upon $M = \dfrac{E}{ME^2} \ldots\ldots\ldots$ in ME,

attraction of S upon $E = \dfrac{m'}{SE^2}$ in ES, equivalent to

$$\left\{\begin{array}{ll} m'\dfrac{EG}{SE^3} \ldots\ldots & \text{in } EG, \\ m'\dfrac{r'}{SE^3} \ldots & \text{parallel to } GS; \end{array}\right.$$

attraction of M upon $E = \dfrac{M}{ME^2} \ldots\ldots\ldots\ldots\ldots\ldots$ in EM.

Therefore, the whole attraction upon M, when E is brought to rest, is

$$\frac{E+M}{ME^2} + m'\left(\frac{MG}{SM^3} + \frac{EG}{SE^3}\right) \text{ in } ME,$$

and

$$m'r'\left(\frac{1}{SM^3} - \frac{1}{SE^3}\right) \text{ parallel to } GS.$$

These expressions of the accelerating forces on M are rigorous, and can be expressed in terms of the masses and coordinates of the bodies; but since our investigations will be carried only to the second order, it will be sufficient if, in the preceding, we neglect small quantities of the fourth and higher orders.

Now,
$$
\begin{aligned}
SM^2 &= SM'^2 + MM'^2 \\
&= SG^2 + GM'^2 - 2SG\ GM' \cos SGM' + MM'^2 \\
&= r'^2\left(1 - 2\,\frac{GM'}{r'}\cos SGM' + \frac{GM'^2}{r'^2}\right);
\end{aligned}
$$

therefore
$$\frac{1}{SM^3} = \frac{1}{r'^3}\left\{1 + \frac{3\,GM'}{r'}\cos(\theta-\theta')\right\};$$

for $\theta-\theta'$ or $SE'M'$ differs from SGM' by less than 7″, Art (17), and $\left(\frac{GM'}{r'}\right)^2$ is neglected, being of the fourth order, Art. (22)

Similarly,
$$\frac{1}{SE^3} = \frac{1}{r'^3}\left\{1 - \frac{3\,GE'}{r'}\cos(\theta-\theta')\right\};$$

therefore, the accelerating forces on the moon are approximately

$$\frac{\mu}{ME^2} + \frac{m'}{r'^3}(MG + GE) \text{ .... in direction } ME,$$

and
$$\frac{3m'}{r'^3}(GM' + GE')\cos(\theta-\theta') \text{ .... parallel to } GS;$$

whence
$$
\begin{aligned}
P &= \left(\frac{\mu}{ME^2} + \frac{m'}{r'^3}ME\right)\cos MGM' - \frac{3m'}{r'^3}M'E'\cos^2(\theta-\theta') \\
&= \frac{\mu}{r^2(1+s^2)^{\frac{3}{2}}} + \frac{m'r}{r'^3} - \frac{3m'r}{2r'^3}\{1+\cos 2(\theta-\theta')\} \\
&= \mu u^2(1 - \tfrac{3}{2}s^2) - \frac{m'u'^3}{u}\{\tfrac{1}{2} + \tfrac{3}{2}\cos 2(\theta-\theta')\}, \\
T &= -\frac{3m'}{r'^3}M'E'\cos(\theta-\theta')\sin(\theta-\theta') \\
&= -\tfrac{3}{2}\frac{m'u'^3}{u}\sin 2(\theta-\theta') \\
S &= \left(\frac{\mu}{ME^2} + \frac{m'}{r'^3}ME\right)\sin MGM' \\
&= \left\{\frac{\mu}{r^2(1+s^2)} + \frac{m'r\sqrt{(1+s^2)}}{r'^3}\right\}\frac{s}{\sqrt{(1+s^2)}} \\
&= \mu u^2(s - \tfrac{3}{2}s^3) + \frac{m'u'^3 s}{u}; \\
Ps - S &= -\frac{m'u'^3 s}{u}\{\tfrac{3}{2} + \tfrac{3}{2}\cos 2(\theta-\theta')\}
\end{aligned}
$$

24 The differential equations in Art (20), when these values of the forces are substituted in them, would contain a new variable θ', but we shall find means to establish a connexion between t, θ, and θ' which will enable us to eliminate θ'.

They will, however, be still incapable of solution except by successive approximation; but before proceeding to this, it will be important to consider the order of the *disturbing* effect of the sun's action, compared with the direct action of the earth Now, if we examine the values of P, T, and S, it will be found that the most important of the terms containing m', which are clearly the disturbing forces since they depend upon the sun, are involved in the form $\frac{m'r}{r'^3}$, while those independent of the sun's action enter in the form $\frac{\mu}{r^2}$.

We must therefore find the order of $\frac{m'r}{r'^3}$ compared with $\frac{\mu}{r^2}$,

$$\text{or of } \frac{m'}{r'^3} \;\ldots\;.\;.\qquad ..\;.\;\frac{\mu}{r^3}$$

Now the orbits being nearly circular, and m the ratio of the mean motions, Art (22), we have

$$m = \frac{\text{mean motion of sun}}{\text{mean motion of moon}} = \frac{\text{periodic time of moon}}{\text{periodic time of sun}}$$

$$= \frac{2\pi r^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} - \frac{2\pi r'^{\frac{3}{2}}}{m'^{\frac{1}{2}}};$$

therefore $$\frac{m'}{r'^3} : \frac{\mu}{r^3} :: m^2 : 1,$$

or the disturbing force of the sun is of the second order.

CHAPTER IV

INTEGRATION OF THE DIFFERENTIAL EQUATIONS.

SECTION I

General process described.

25. The differential equations which we have obtained are, as already stated, incapable of solution in their general forms; and even when P, T, &c have been replaced by their values, the integration cannot be effected, and we must proceed by successive approximation.

Firstly, neglect the disturbing force of the sun which is of the second order, and also the moon's latitude, which, as will be seen by referring to the expressions for the forces (23), will either enter to the second power or else in combination with the disturbing force.

When this is done the equations can be integrated, and values of u and s obtained in terms of θ correct to the same order of approximation as the differential equations themselves, that is, to the first order, and this value of u will enable us to get the connexion between θ and t to the same order

[Let us, however, bear in mind that the equations thus integrated are not the differential equations of the moon's motions, but only approximate forms of them, and it is, therefore, possible that the results obtained may not be even approximate forms of the true solutions

Whether they are so or not, can only appear by comparing them with what we already know of the motion from observation, and this previous knowledge, in the event of their not being approximations, will probably suggest such modifications of them as will render them so *]

The integration of the equations (α), (β), (γ) can be performed when the second members are circular functions of θ; and as the first approximation will give us the values of u and s in that form, when these values are admissible and carried into the expressions for the forces, they also will be expressed as functions of θ, and we can proceed to a higher approximation.

The new approximate values of P, T, S are then made use of to reduce the second members of the differential equations to functions of θ, retaining those terms of the expressions which are of the second order

The equations are again integrable, and this being done, the values of u, s, t will be obtained correctly to the second order These values introduced in the same manner in the second members, and terms of the next higher order retained, will lead to a third approximation, and so on, to any order, except that if we wish to carry it on beyond the third, the approximate values of the forces, given in Art (23), would no longer be sufficiently exact

26. There is, however, a peculiarity in these equations, when solved by this process, which we must notice We have said that to obtain the values to any order, all terms up to that order must be retained in the second members but it may happen that a term of an order beyond that to which we are working would, if retained, be so altered by the integration as to come within the proposed order.

Such terms must therefore not be rejected, and we shall proceed to examine by what means they may be recognised.

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* We cannot say *à priori* that the solution of $\frac{d^2u}{d\theta^2} + u = f(u, \theta)$ shall be of the same form as that of $\frac{d^2u}{d\theta^2} + u = \phi(\theta)$, even though $\phi(\theta)$ should be a very approximate value of $f(u, \theta)$, but there is a presumption in favour of such a supposition

27. Suppose then that after an approximation to a certain order, the substitutions for the next step have brought the equation in u to the form

$$\frac{d^2u}{d\theta^2} + u = \ldots\ldots + G\cos(p\theta + H) + \ldots\ldots ,$$

where the coefficient G is one order beyond that which we intend to retain. The solution of this equation will be of the form

$$u = \ldots\ldots + G'\cos(p\theta + H) + \ldots\ldots,$$

G' being a constant to be determined by putting this value of u in the differential equation,

whence $$G' = \frac{G}{1-p^2},$$

from which we learn that if p differs very little from 1, G' will be one order *lower* than G, and will come within our proposed approximation, and consequently the term $G\cos(p\theta + H)$ must be retained in the differential equation.

The equation of the moon's latitude being of the same form as that of the radius vector, the same remarks apply to it.

28. Again, in finding the connexion between the longitude and the time (one of the principal objects of the Theory), we must use equation (α), Art. (20),

$$\frac{dt}{d\theta} = \frac{1}{hu^2\sqrt{\left(1 + 2\int\frac{T}{h^2u^3}\,d\theta\right)}}.$$

Now, having developed the second member and substituted for u, &c. their values in terms of θ, let it become

$$\frac{dt}{d\theta} = \ldots\ldots + Q\cos(q\theta + R) + \ldots\ldots;$$

hence, $$t = \ldots\ldots + \frac{Q}{q}\sin(q\theta + R) + \ldots\ldots$$

Therefore, when q is of the first order, $\frac{Q}{q}$ will be one order *lower* than Q, and the term will have risen in importance by the integration.

But yet further, if such terms occur in $\frac{T}{u^3}$, they will be twice increased in value; for they increase once in forming $\int \frac{T}{h^2u^3}\,d\theta$, and once again, as above, in finding t.

Since such terms occur in the development of $\frac{T}{u^3}$, and also of $\frac{d\theta}{dt}$, on account of their previously being found in u, we must examine how they appear in the differential equation that gives u, that we may recognise and retain them at the outset. Now, by referring to the last article, we see that when p is very small G and G' will be of the same order; and in $\frac{1}{u^2}$, $\frac{T}{u^3}$ the order of the term will still be the same; so that all the terms which it will be necessary to retain are known at the outset.

29. We have, therefore, the following rule:

In approximating to any given order, we must, in the differential equations for u *and* s, *retain periodical terms* ONE ORDER *beyond the proposed one, when the coefficient of* θ *in their argument is nearly equal to* 1 *or* 0; *and terms in which the coefficient of the argument is nearly equal to* 0, *must be retained* TWO ORDERS *higher than the proposed approximation when they occur in* $\frac{\mathrm{T}}{\mathrm{h}^2\mathrm{u}^3}$.

If we wished to obtain u only and not t, there would be no necessity for retaining those terms of a more advanced order in which the coefficient of θ nearly equals 0.

SECTION II.

To solve the Equations to the first order

30. We shall in this first step neglect the terms which depend on the disturbing force, *i e* those terms which contain m', for we have seen, Art. (24), that such terms will be of the second order

The differential equations may be written under the more convenient form

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}-\frac{T}{h^2u^3}\frac{du}{d\theta}-2\left(\frac{d^2u}{d\theta^2}+u\right)\int\frac{T}{h^2u^3}\,d\theta\;.\;(\beta'),$$

$$\frac{d^2s}{d\theta^2}+s=\frac{Ps-S}{h^2u^3}-\frac{T}{h^2u^3}\frac{ds}{d\theta}-2\left(\frac{d^2s}{d\theta^2}+s\right)\int\frac{T}{h^2u^3}\,d\theta\;.\;(\gamma')$$

The latitude s of the moon can never exceed the inclination of the orbit to the ecliptic, but this inclination is of the first order, therefore s is at least of the first order and s^2 may be neglected

Therefore, from Art (23),

$$\frac{P}{u^2}=\mu;\quad \frac{T}{u^3}=0;\quad \frac{Ps-S}{u^3}=0,$$

and the differential equations become

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2},$$

$$\frac{d^2s}{d\theta^2}+s=0;$$

whence $u=\frac{\mu}{h^2}\{1+e\cos(\theta-\alpha)\}$, or, writing a for $\frac{\mu}{h^2}$,

$$u=a\{1+e\cos(\theta-\alpha)\}\quad\ldots\ldots\quad(U_1),$$

and $s=k\sin(\theta-\gamma)\quad\ldots\ldots\quad(S_1)$,

e, α, k, γ being the four constants introduced by integration

31. These results are in perfect agreement with what *rough* observations had already taught us concerning the moon's motion Art. (22), for

$$u=a\{1+e\cos(\theta-\alpha)\}$$

represents motion in an ellipse about the earth as focus.

Again, $s=k\sin(\theta-\gamma)$ indicates motion in a plane inclined to the ecliptic at an angle $\tan^{-1}k$.

For, if $\Upsilon OM'$ be the ecliptic, (fig 7)

M the moon's place,

MM' an arc perpendicular to the ecliptic,

then $\Upsilon M'=\theta$,

and if ΥO be taken equal to γ, and OM joined by an arc of great circle, we have

$$\sin OM' = \tan MM' \cot MOM';$$

or

$$\sin(\theta-\gamma) = s \cot MOM',$$

which, compared with the equation above, shews that

$$MOM' = \tan^{-1}k$$

Therefore, the moon is in a plane passing through a fixed point O and making a constant angle with the ecliptic; or, the moon moves in a plane.

32 What the equations can not teach us, however, and for which we must have recourse to our observations, is the approximate magnitude of the quantities e and k By referring to Art. (22), we see that e is about $\frac{1}{20}$ and k about $\frac{1}{12}$, that is, both these quantities are of the first order Their exact values cannot yet be obtained the means of doing so from multiplied observations will be indicated further on.

The values of α and γ introduced in the above solutions are respectively the longitude of the apse and of the node

33. Lastly, to find the connexion between t and θ, the equation (α) becomes, making $T=0$,

$$\frac{dt}{d\theta} = \frac{1}{hu^2} = \frac{1}{ha^2} \frac{1}{\{1+e\cos(\theta-\alpha)\}^2}$$

Now this is the very same equation that we had connecting t and θ in the problem of *two bodies*, Art. (12), as we ought to expect, since we have neglected the sun's action Therefore, if p be the moon's mean angular velocity, we should, following the same process as in the article referred to, arrive at the result

$$\theta = pt + \varepsilon + 2e\sin(pt+\varepsilon-\alpha) + \tfrac{5}{4}e^2\sin 2(pt+\varepsilon-\alpha) + \ldots\ldots,$$

which is correct only to the first order, since we have rejected some terms of the second order by neglecting the disturbing force

34 The arbitrary constant ε, introduced in the process of integration, can be got rid of by a proper assumption this assumption is, that the time t is reckoned from the instant when the *mean* value of θ is zero *

For, since the *mean* value of θ, found by rejecting the periodical terms, is $pt + \varepsilon$; if, when this vanishes, $t = 0$, we must have $\varepsilon = 0$; therefore

$$\theta = pt + 2e \sin(pt - \alpha) \quad . \qquad . \ . \ . \ . \Theta_1,$$

correct to the first order †

35 We have now obtained three results, U_1, S_1, Θ_1, as solutions to the first order of our differential equations, and we must employ them to obtain the next approximate solutions. but before U_1 and S_1 can be so employed they must be slightly modified, in such a manner however as not to interfere with their degree of approximation

The necessity for such a modification will appear from the following considerations.

Suppose we proceed with the values already obtained; we have, by Art. (23),

$$\frac{P}{h^2u^2} = \frac{\mu}{h^2}\left(1 - \tfrac{3}{2}s^2\right) - \frac{m'u'^3}{2h^2u^3} - \&c. \quad ..$$

$$= a\left(1 - \tfrac{3}{2}s^2\right) - \frac{m'u'^3}{2h^2a^3}\{1 + e\cos(\theta - \alpha)\}^{-3},$$

$$= . \ . \qquad . \quad + A\cos(\theta - \alpha) + \ . \ ;$$

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* When a function of a variable contains periodical terms which go through all their changes positive and negative as the variable increases continuously, the *mean value* of the function is the part which is independent of the periodical terms

† We shall also employ this method of correcting the integral in our next approximation to the value of θ in terms of t, and if we purposed to carry our approximations to a higher order than the second, we should still adopt the same value, that is, zero, for the arbitrary constant introduced by the integration To shew the advantage of thus correcting with respect to *mean* values suppose we reckoned the time from some *definite* value of θ, for instance when $\theta = 0$, then, in the first approximation,

$$0 = \varepsilon + 2e\sin(\varepsilon - \alpha),$$

or substituting for $\frac{P}{h^2u^2}$ from Art (45),

$$\frac{d^2u}{d\theta^2}+u=a\left\{\begin{aligned}&1-\tfrac{3}{4}k^2-\tfrac{1}{2}m^2+\tfrac{3}{2}m^2e\cos(c\theta-\alpha)-\tfrac{3}{2}m^2\cos\{(2-2m)\theta-2\beta\}\\&\qquad\qquad\qquad+\tfrac{9}{4}m^2e\cos\{(2-2m-c)\,\theta-2\beta+\alpha\}\\&\qquad\qquad\qquad+\tfrac{3}{4}k^2\cos 2(g\theta-\gamma)-\tfrac{3}{2}m^2e'\cos(m\theta+\beta-\zeta);\end{aligned}\right.$$

therefore,

$$u=a\left\{\begin{aligned}&1-\tfrac{3}{4}k^2-\tfrac{1}{2}m^2+e\cos(c\theta-\alpha)+\tfrac{1}{2}m^2\cos\{(2-2m)\theta-2\beta\}\\&\qquad\qquad\qquad+\tfrac{9}{16}me\cos\{(2-2m-c)\,\theta-2\beta+\alpha\}\\&\qquad\qquad\qquad-\tfrac{1}{4}k^2\cos 2(g\theta-\gamma)-\tfrac{3}{2}m^2e'\cos(m\theta+\beta-\zeta)\end{aligned}\right.$$

If we compare this with the value of u found Art (48), we see that the elliptic inequality, the reduction, and the annual equation are due to the central or radial force, as also one half of the variation and about a third of the evection

It would perhaps be proper to separate the absolute central force from the central *disturbing* force, the terms due to the latter are those which contain m, therefore, the elliptic inequality and the reduction are the effects of the former, except that in the elliptic inequality the introduction of c, or the motion of the apse, is due to the disturbing force

(2) *To determine the effect of the tangential disturbing force*

Let the central *disturbing* force be zero,

then $$\frac{P}{h^2u^2}=\frac{\mu}{h^2}\,(1-\tfrac{3}{2}s^2)=a \text{ neglecting the inclination,}$$

$$\frac{T}{h^2u^3}=-\tfrac{3}{2}m^2\sin\{(2-2m)\,\theta-2\beta\}$$
$$+\,3m^2e\sin\{(2-2m-c)\,\theta-2\beta+\alpha\}$$

omitting the term of the fourth order,

$$u=a\{1+e\cos(c\theta-\alpha)\};$$

therefore $$\frac{T}{h^2u^3}\frac{du}{d\theta}=\tfrac{3}{4}m^2ae\cos\{(2-2m-c)\,\theta-2\beta+\alpha\},$$

$$\int\frac{T}{h^2u^3}\,d\theta=\tfrac{3}{4}m^2\cos\{(2-2m)\,\theta-2\beta\}$$
$$-\,3m^2e\cos\{(2-2m-c)\,\theta-2\beta+\alpha\},$$

$$\frac{d^2u}{d\theta^2}+u=a \text{ to the first order}$$

Substituting these values in the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} - \frac{T}{h^2u^3}\frac{du}{d\theta} - 2\left(\frac{d^2u}{d\theta^2} + u\right)\int\frac{T}{h^2u^3}\,d\theta$$

$$= a\begin{cases}1 - \frac{3}{2}m^2\cos\{(2-2m)\,\theta - 2\beta\} \\ \quad + \frac{21}{4}m^2e\cos\{(2-2m-c)\,\theta - 2\beta + \alpha\},\end{cases}$$

whence $$u = a\begin{cases}1 + e\cos(c\theta - \alpha) + \frac{1}{2}m^2\cos\{(2-2m)\,\theta - 2\beta\} \\ \quad + \frac{21}{16}me\cos\{(2-2m-c)\,\theta - 2\beta + \alpha\}\end{cases}$$

We have here the remaining half of the variation and rather more than two-thirds of the evection as the effects of the tangential disturbance Also $c = 1$, or, to the second order, the tangential force has no effect on the motion of the apse.

The inequalities in the longitude could be easily obtained from the relation

$$\frac{dt}{d\theta} = \frac{1}{hu^2\left(1 + 2\int\frac{T}{h^2u^3}\,d\theta\right)^{\frac{1}{2}}},$$

but they would lead to the very same conclusions as the discussion of the values of u

To calculate the value of c to the third order

94 We must here make use of the results which the approximations to the second order have furnished; but as the value of c is determined by that term of the differential equation whose argument is $c\theta - \alpha$, we need only consider those terms which by their combinations will lead to that one without rising to a higher order than the fourth.

We shall simplify the arguments by omitting θ, α, β, which can easily be supplied by remarking that $c\theta - \alpha$ and $m\theta - \beta$ always enter as one symbol, c and m will therefore be sufficient to distinguish them This only applies to the arguments.

We have, Arts (48), (23),

$$u = a\{1 + e\cos(c) + m^2\cos(2-2m) + \tfrac{15}{8}me\cos(2-2m-c)\},$$

$$\frac{P}{h^2u^2} = a - \tfrac{1}{2}m^2\,\frac{a^4}{u^3}\,\{1 + 3\,\cos(2-2m)\},$$

$$\frac{T}{h^2u^3} = \qquad - \tfrac{3}{2}m^2\,\frac{a^4}{u^4}\,\sin(2-2m)$$

From these, we obtain

$$\begin{aligned}\frac{P}{h^2u^2} &= a - \tfrac{1}{2}m^2a\,\{1 + 3\,\cos(2-2m)\}\,\{1 - 3e\,\cos(c) \\ &\qquad\qquad\qquad + \qquad - \tfrac{45}{8}me\,\cos(2-2m-c)\} \\ &= a + \tfrac{3}{2}m^2ae\,\cos(c) + \tfrac{135}{32}m^3ae\,\cos(c),\end{aligned}$$

$$\begin{aligned}\frac{T}{h^2u^3} &= -\tfrac{3}{2}m^2\,\sin(2-2m)\,\{1 - 4e\,\cos(c) - 4m^2\cos(2-2m) \\ &\qquad\qquad\qquad - \tfrac{15}{2}me\,\cos(2-2m-c)\} \\ &= -\tfrac{3}{2}m^2\sin(2-2m) + 3m^2e\sin(2-2m-c) + \tfrac{45}{8}m^3e\sin(c),\end{aligned}$$

$$\begin{aligned}\frac{T}{h^2u^3}\,\frac{du}{d\theta} &= \{-\tfrac{3}{2}m^2\,\sin(2-2m)\}\,\{-\tfrac{15}{8}mae\,\sin(2-2m-c)\} \\ &\qquad + \{3m^2e\,\sin(2-2m-c)\}\,\{-2m^2a\,\sin(2-2m)\} \\ &= \tfrac{45}{32}m^3ae\,\cos(c),\end{aligned}$$

the other term is of the fifth order,

$$\int\frac{T}{h^2u^3}\,d\theta = \tfrac{3}{4}m^2\,\cos(2-2m) - 3m^2e\,\cos(2-2m-c) - \tfrac{45}{8}m^3e\cos(c),$$

$$2\left(\frac{d^2u}{d\theta^2} + u\right) = 2a\{1 + \quad - 3m^2\,\cos(2-2m) + \tfrac{15}{2}m^2e\cos(2-2m-c)\};$$

therefore $2\left(\dfrac{d^2u}{d\theta^2} + u\right)\displaystyle\int\frac{T}{h^2u^3}\,d\theta = \quad . \quad - \tfrac{45}{4}m^3ae\,\cos(c).$

Substituting in the equation for u,

$$\begin{aligned}\frac{d^2u}{d\theta^2} + u &= \frac{P}{h^2u^2} - \frac{T}{h^2u^3}\,\frac{du}{d\theta} - 2\left(\frac{d^2u}{d\theta^2} + u\right)\int\frac{T}{h^2u^3}\,d\theta \\ &= a\{1 + (\tfrac{3}{2}m^2e + \tfrac{135}{32}m^3e - \tfrac{45}{32}m^3e + \tfrac{45}{4}m^3e)\,\cos(c) + \,..\,\} \\ &= a\{1 + (\tfrac{3}{2}m^2e + \tfrac{225}{16}m^3e)\,\cos(c) + \ .\ \}.\end{aligned}$$

Assume $u = a\{1 + e\,\cos(c) + \ .\ \}$;

therefore $ae(1-c^2) = (\tfrac{3}{2}m^2e + \tfrac{225}{16}m^3e)\,a$;

therefore $c = 1 - \tfrac{3}{4}m^2 - \tfrac{225}{32}m^3.$

To find the value of g *to the third order*

95 This is to be obtained in a very similar manner from the equation $\frac{d^2s}{d\theta^2} + s = \&c.$ We shall, in the argument, write g for $g\theta - \gamma$

$$s = k\{\sin(g) + \tfrac{3}{8}m \sin(2 - 2m - g)\},$$

$$\frac{Ps - S}{h^2u^3} = -\frac{3m^2a^4s}{2u^4}\{1 + \cos(2 - 2m)\},$$

$$\frac{T}{h^2u^3} = -\frac{3m^2a^4}{2u^4}\sin(2 - 2m)$$

Whence

$$\frac{Ps - S}{h^2u^3} = -\tfrac{3}{2}m^2k\{1 + \cos(2 - 2m)\}\{\sin(g) + \tfrac{3}{8}m \sin(2 - 2m - g)\}$$

$$= \qquad -\tfrac{3}{2}m^2k\,(1 - \tfrac{3}{16}m)\sin(g),$$

$$\frac{ds}{d\theta} = k\cos(g) + \tfrac{3}{8}mk\cos(2 - 2m - g);$$

therefore $\frac{T}{h^2u^3}\frac{ds}{d\theta} = \qquad -\tfrac{9}{32}m^3k\sin(g),$

$$\frac{d^2s}{d\theta^2} + s = \text{terms of the third order};$$

therefore $2\left(\frac{d^2s}{d\theta^2} + s\right)\int\frac{T}{h^2u^3}\,d\theta = 0$, to the fourth order.

Now, $$\frac{d^2s}{d\theta^2} + s = \frac{Ps - S}{h^2u^3} - \frac{T}{h^2u^3}\frac{ds}{d\theta} - 2\left(\frac{d^2s}{d\theta^2} + s\right)\int\frac{T}{h^2u^3}\,d\theta$$

$$= -\tfrac{3}{2}m^2k\,(1 - \tfrac{3}{16}m - \tfrac{3}{16}m)\sin(g).$$

Assume $s = k\sin(g)$;

therefore $k(1 - g^2) = -\tfrac{3}{2}m^2k + \tfrac{9}{16}m^3k,$

$$g = 1 + \tfrac{3}{4}m^2 - \tfrac{9}{32}m^3$$

96. Hence, to the third order of approximation,

$$\frac{\text{mean motion of apse}}{\text{mean motion of node}} = \frac{1 - c}{g - 1} = \frac{\tfrac{3}{4}m^2 + \tfrac{225}{32}m^3}{\tfrac{3}{4}m^2 - \tfrac{9}{32}m^3} = \frac{8 + 75m}{8 - 3m},$$

and since $m = \frac{1}{13}$ nearly, we see that the moon's apse progredes nearly twice as fast as the node regredes

In the case of one of Jupiter's satellites, m is extremely small, for the periodic time round Jupiter is only a few of our days, and the periodic time of Jupiter round the sun is 12 of our years, and therefore m, the ratio of these periods, is very small.

Hence, the apse of one of Jupiter's satellites progredes along Jupiter's ecliptic, with pretty nearly the same velocity as the node regredes, assuming these motions to be due to the sun's disturbing force, they are, however, principally due to the oblateness of the planet.

Parallactic Inequality

97. In carrying on the approximations to a higher order, it is found, as we stated Art (55), that the expressions for $\frac{P}{h^2u^2}$ and $\frac{T}{h^2u^3}$ contain the terms $-\frac{9}{8}m^2a\,\frac{E-M}{E+M}\,\frac{a'}{a}\cos(\theta-\theta')$, and $-\frac{3}{8}m^2\,\frac{E-M}{E+M}\,\frac{a'}{a}\sin(\theta-\theta')$ respectively

Since $\frac{a'}{a} = \frac{1}{400}$, nearly, is of the second order, these terms are of the fourth order, but the coefficient of θ being near unity, they will become important in u, and therefore in θ, Art (27).

We can easily obtain the terms to which they give rise in the values of u and θ,

$$\frac{P}{h^2u^2} = a\left[1 + \qquad -\frac{9}{8}m^2\,\frac{E-M}{E+M}\,\frac{a'}{a}\cos\{(1-m)\,\theta-\beta\}\right],$$

$$\frac{T}{h^2u^3} = \quad . \qquad . \quad -\frac{3}{8}m^2\,\frac{E-M}{E+M}\,\frac{a'}{a}\sin\{(1-m)\,\theta-\beta\},$$

$$u = a\{1 - e\cos(c\theta-\alpha) + \ .\ .\ \},$$

$$\frac{du}{d\theta} = ae\sin(c\theta-\alpha)\ .\ .\ ;$$

therefore

$$\frac{T}{h^2u^3}\frac{du}{d\theta} = 0, \text{ to the fourth order,}$$

$$\frac{d^2u}{d\theta^2} + u = a, \text{ to the second order,}$$

$$\int \frac{T}{h^2u^3}\,d\theta = \;. \qquad + \tfrac{3}{8}m^2\frac{E-M}{E+M}\frac{a'}{a}\cos\{(1-m)\,\theta - \beta\}$$

Substituting in the differential equation for u, we get

$$\frac{d^2u}{d\theta^2} + u = a\left[1 + \;.\,. \quad - \tfrac{15}{8}m^2\frac{E-M}{E+M}\frac{a'}{a}\cos\{(1-m)\,\theta - \beta\} + \qquad\right]$$

Assume $u = a\left[1 + \;. \qquad + A\,\dfrac{E-M}{E+M}\dfrac{a'}{a}\cos\{(1-m)\,\theta - \beta\} + . \quad\right];$

therefore $$A = \frac{-\tfrac{15}{8}m^2}{1-(1-m)^2} = -\tfrac{15}{16}m;$$

therefore $u = a\left[1 + \;. \qquad - \tfrac{15}{16}m\,\dfrac{E-M}{E+M}\dfrac{a'}{a}\cos\{(1-m)\theta - \beta) + \qquad\right]$

98. The corresponding term in the value of θ will also be of the third order,

$$\frac{dt}{d\theta} = \frac{1}{hu^2}\left(1 - \int\frac{T}{h^2u^3}\,d\theta\right),$$

$$\frac{1}{u^2} = \frac{1}{a^2}\left[1 + \quad .\,. + \tfrac{15}{8}m\,\frac{E-M}{E+M}\frac{a'}{a}\cos\{(1-m)\,\theta - \beta\} + \qquad\right],$$

$\displaystyle\int\frac{T}{h^2u^3}\,d\theta$ is of the fourth order and will not rise in t;

therefore

$$\frac{dt}{d\theta} = \frac{1}{p}\left[1 + . \;\; .\,. + \tfrac{15}{8}m\,\frac{E-M}{E+M}\frac{a'}{a}\cos\{(1-m)\,\theta - \beta\} + \;.\right],$$

$$t = \frac{1}{p}\left[\theta + \quad . + \tfrac{15}{8}m\,\frac{E-M}{E+M}\frac{a'}{a}\sin\{(1-m)\,\theta - \beta\} + .\,.\right],$$

and $\theta = \;pt\; + . \,. \,. - \tfrac{15}{8}m\,\dfrac{E-M}{E+M}\dfrac{a'}{a}\sin\{(1-m)pt - \beta\} + . \,.$

This term, whose argument is the angular distance of the sun and moon, is called the *parallactic inequality* on account of its use in the determination of the sun's parallax, to which purpose it was first applied by Mayer by comparing the analytical expression of this coefficient with its value as deduced from observation. The values of m and of $\frac{E}{M}$, and therefore, of $\frac{E-M}{E+M}$ being pretty accurately known, $\frac{a'}{a}$ will be determined, that is, the ratio of the sun's parallax to that of the moon; but the moon's parallax is well known; therefore also, that of the sun can be calculated. The value so obtained for the sun's parallax is 8·63221″, while those given by the two last transits of Venus fall between 8·5″ and 8·7″.*

Secular Acceleration

99. Halley, by the comparison of ancient and modern eclipses, found that the moon's mean revolution is now performed in a shorter time than at the epoch of the recorded Chaldean and Babylonian eclipses. The explanation of this phenomenon, called the *secular acceleration of the moon's mean motion*, was for a long time unknown: it was at last satisfactorily given by Laplace.

The value of p, Art. (50), on which the length of the mean period depends, is found, when the approximation is carried to a higher order, to contain the quantity e' the eccentricity of the earth's orbit. Now, this eccentricity is undergoing a slow but continual change from the action of the planets, and therefore p, as deduced from observations made in different centuries, will have different values.

The value of p is at present increasing, or the mean motion is being accelerated, and it will continue thus to increase for a period of immense, but not infinite duration; for, as shewn by

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* Pontecoulant, *Système du Monde*, vol. iv. p. 606.

Lagrange, the actions of the planets on the eccentricity of the earth's orbit will be ultimately reversed, e' will cease to diminish and begin to increase, and consequently p will begin to decrease, and the *secular acceleration* will become a *secular retardation*

It is worthy of remark that the action of the planets on the moon, thus transmitted through the earth's orbit, is more considerable than their direct action

Inequalities depending on the Figure of the Earth

100 The earth, not being a perfect sphere, will not attract as if the whole of its mass were collected at its centre hence, some correction must be introduced to take into account this want of sphericity, and some relation must exist between the oblateness and the disturbance it produces Laplace in examining its effect found that it satisfactorily explained the introduction of a term in the longitude of the moon, which Mayer had discovered by observation, and the argument of which is the true longitude of the moon's ascending node.

By a comparison of the observed and theoretical values of the coefficient of this term, we may determine the oblateness of the earth with as great, if not greater, accuracy than by actual measures on the surface

101 By pursuing his investigations, with reference to the oblateness, in the expression for the moon's latitude, Laplace found that it would there give rise to a term in which the argument was the true longitude of the moon

This term, which was unsuspected before, will also serve to determine the earth's oblateness, and the agreement with the result of the preceding is almost perfect, giving the compression $\frac{1}{305}$,* which is about a mean between the different values obtained by other methods

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* Pontecoulant, *Système du Monde*, vol IV

Perturbations due to Venus

102 After the expression for the moon's longitude had been obtained by theory, it was found that there was still a slight deviation between her calculated and observed places, and Burg, who discovered it by a discussion of the observations of Lahire, Flamsteed, Bradley, and Maskelyne, thought it could be represented by an inequality whose period would be 184 years and coefficient 15″. This was entirely conjectural, and though several attempts were made, it was not accounted for by theory

About five years ago, Professor Hansen, of Seeberg in Gotha, having commenced a revision of the Lunar Theory, found two terms, which had hitherto been neglected, due to the action of Venus One of them is direct and arises from a 'remarkable 'numerical relation between the anomalistic motions of the moon 'and the sidereal motions of Venus and the earth; the other is 'an indirect effect of an inequality of long period in the motions 'of Venus and the earth, which was discovered some years ago 'by the Astronomer Royal'*

The periods of these two inequalities is extremely long, one being 273 and the other 239 years, and their coefficients are respectively 27 4″ and 23 2″. 'These are considerable quantities 'in comparison with some of the inequalities already recognised 'in the moon's motion, and, when applied, they are found to 'account for the chief, indeed the only remaining, empirical 'portion of the moon's motion in longitude of any consequence, 'so that their discovery may be considered as a practical completion of the Lunar Theory, at least for the present astronomical age, and as establishing the entire dominion of the 'Newtonian Theory and its analytical application over that 'refractory satellite.'†

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* Report to the Annual General Meeting of the Royal Astronomical Society, Feb 11, 1848.

† Address of Sir John Herschel to the Meeting of the Royal Astronomical Society

Motion of the Ecliptic

103 We have seen, Art. (14), that our plane of reference is not a fixed plane, but its change of position is so slow that we have been able to neglect it, and it is only when the approximation is carried to a high order, that the necessity arises for taking account of its motion

It has been found to have an angular velocity, about an axis in its own plane, of 48″ in a century, and the correction thus introduced produces in the latitude of the moon a term

$$-c\omega\cos(\theta-\phi),$$

where ω is the angular velocity of the ecliptic, $\frac{1}{c}$ the angular velocity with which the ascending node of the moon's orbit recedes from the instantaneous axis about which the ecliptic rotates, ϕ the longitude of this axis at time t, and θ the longitude of the moon at the same instant

Let ΥAM (fig. 11½) be the position of the ecliptic at time t,
A the point about which it is turning, $\Upsilon A=\phi$,
NM the moon's orbit, M the moon, and Mm a perpendicular to the ecliptic; $\Upsilon m=\theta$, $Mm=\text{lat}=\beta$.

Let $APN'm'$ be the ecliptic after a time δt

Any point whose longitude is L may be considered as moving perpendicularly to the ecliptic with a velocity $\omega\sin(L-\phi)$

Hence, if i be the inclination of the orbit, and N the longitude of the node, the point N will move in the direction NP with a velocity $\omega\sin(N-\phi)$ And N' will move along PN' with a velocity $\omega\sin(N-\phi)\cot i$,

therefore $$\frac{dN}{dt}=\omega\sin(N-\phi)\cot i.$$

Again, the point of the ecliptic 90° in advance of N, will move towards the moon's orbit with a velocity $\omega\sin(90+N-\phi)$;

therefore $$\frac{di}{dt}=-\omega\cos(N-\phi).$$

Now, $\cot i$, ω, and $\frac{d(N-\phi)}{dt}$ may be considered constant in integrating;

therefore $$\delta N = c\omega \cos(N-\phi) \cot i,$$
$$\delta i = c\omega \sin(N-\phi),$$
where $$\frac{d(N-\phi)}{dt} = -\frac{1}{c},$$
and if $NM = \psi$, we have
$$\delta\psi = -\frac{\delta N}{\cos i} = -\frac{c\omega}{\sin i}\cos(N-\phi).$$

Now, $$\sin\beta = \sin i \ \sin\psi;$$
$$\cos\beta \,.\, \delta\beta = \cos i \ \sin\psi \ \delta i + \sin i \ \cos\psi \ \delta\psi$$
$$= c\omega\{\cos i \sin\psi \sin(N-\phi) - \cos\psi \cos(N-\phi)\},$$
but $\cos i \ \sin\psi = \cos\beta \sin(\theta - N)$ and $\cos\psi = \cos\beta\cos(\theta - N)$.
therefore $$\delta\beta = -\, c\omega \cos(\theta - \phi)$$

The discovery of this term is due to Professor Hansen, its coefficient is extremely small, about 1 5″; but, being of a totally different nature from those due to successive approximations, it was thought desirable to examine it, and the above investigation, which was communicated to me by J. C Adams, Esq , will be read with interest on account of its elegance and simplicity

We may obtain an approximate value of the coefficient $c\omega$ by substituting for it $\frac{n\alpha}{2\pi}$, where α is the number of seconds through which the ecliptic is deflected in one year = 0 48″, and n is the number of years in which the node of the moon's orbit makes a complete revolution = 18 6; for then, $\frac{2\pi}{n}$ is the angle described by the node in one year, therefore, $\frac{n\alpha}{2\pi}$ is the ratio of ω $\frac{d(N-\phi)}{dt}$, supposing ϕ to remain constant, which is nearly the case, therefore,
$$c\omega = \frac{n\alpha}{2\pi} = \frac{9\ 3 \times 0\ 48''}{3\ 14} = 1\ 42''.^{*}$$

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* This affords the solution of a problem proposed in the Senate-House in the January Examination of 1852 Question 21, Jan 22

Note on the Numerical Values of the Coefficients

104 When the periods of two of the terms, in the third method given in Art (62), differ but slightly, for instance if θ and ϕ go through their periodic variations very nearly in the same time, the method could not then with safety be applied; for, since the same values of θ and ϕ would very nearly recur together during a longer time than that through which the observations would extend, the two terms would be so blended in the value of V that they would enter nearly as one term—the difference between θ and ϕ would be very nearly the same at the end as at the beginning of the series of observations.

105 Let us suppose the periods to be actually identical, so that $\phi = \theta + \alpha$, α being some constant angle; then

$$B \sin\theta + C \sin\phi$$

may be written $(B + C\cos\alpha)\sin\theta + C\sin\alpha\cos\theta$,

or $\qquad V = A + (B + C\cos\alpha)\sin\theta + C\sin\alpha\cos\theta +$

If now we divide the observations, as before, into two sets, corresponding to the positive and negative values of $\sin\theta$, the terms involving $\cos\theta$ in each set will be as often positive as negative, and will disappear in the summation of each set, and, following the process of the method, will give

$$B + C\cos\alpha = M \text{ suppose.}$$

Dividing again into two sets corresponding to the positive and negative values of $\cos\theta$, the terms in $\sin\theta$ will be cancelled, and the same process will give

$$C\sin\alpha = N \text{ suppose.}$$

Treating in the same way with respect to the angle ϕ, we get two results,

$$C + B\cos\alpha = M',$$

$$-B\sin\alpha = N',$$

from these four equations we easily get

$$B = \frac{M'N + MN'}{M^2 - M'^2} N' \text{ or } = - \frac{M'N + MN'}{N^2 - N'^2} N',$$

$$C = - \frac{N}{N'} B,$$

M, N, M', N' are connected by the equation of condition,

$$M^2 - M'^2 = N'^2 - N^2$$

When the periods of θ and ϕ are nearly, but not exactly, the same, this equation of condition will not hold, and the preceding values of B and C would not be exactly correct, but yet they would be very approximate, especially if the mean between the two values of B be taken

106 We may also, after having taken one of these slightly erroneous values for B, make a further correction by establishing as it were a counterbalancing error in the value of C Let B' be the value so found for B, then, from the V of each of the observations subtract the value $B' \sin\theta$, the result U will be very nearly equal to $A + C\sin\phi +$&c, and from the n equations

$$U_1 = A + C\sin\phi_1 +$$
$$U_2 = A + C\sin\phi_2 +$$
$$U_n = A + C\sin\phi_n +$$

a value C' of C will be obtained, by the rule of Art (62), which will be very approximate, and, at the same time, agree better with B' in satisfying the equations than C itself would do

107 When two terms whose periods are nearly equal do occur, it is plain, by examining the values of M and M', that the errors which would be committed by following the rule, without taking account of this peculiarity, would be the taking $B + C\cos\alpha$ and $C + B\cos\alpha$ for B and C respectively

CHAPTER VIII.

HISTORY OF THE LUNAR PROBLEM BEFORE NEWTON.

108 The idea which most probably suggested itself to the minds of those men who first considered the motion of the moon among the stars, was that this motion is uniform and circular about the earth as a centre

This first result is represented in our value of the longitude by neglecting all small terms and writing $\theta = pt$.

109. It must, however, have been very soon perceived that the actual motion is far from being so simple, and that the moon moves with very different velocities at different times.

The earliest recorded attempts to take into account the irregularities of the moon's motion were made by Hipparchus, (140 B C) He imagined the moon to move with uniform velocity in a circle, of which the earth occupied, not the centre, but a point nearer to one side. By a similar hypothesis he had accounted for the irregularities in the sun's motion, and his success in this led him to apply it also to the moon.

It is clear that, on this supposition, the moon would seem to move faster when nearest the earth or in perigee, and slower when in apogee, than at any other points of her orbit, and thus an apparent unequal motion would be produced

Let *BAM* (fig. 12) be a circle, *CA* a radius, *E* a point in *AC* near *C*; *CB*, *ED* two parallel lines making an angle α with *CA*.

Suppose a body *M* to describe this circle uniformly with an angular velocity *p*, the time being reckoned from the instant when the body was at *B*, and the longitude as seen from *E* being reckoned from the line *ED*,

therefore
$$DEM = \theta, \qquad BCM = pt,$$
$$AEM = \theta - \alpha, \quad ACM = pt - \alpha.$$

Now $\frac{EC}{CM}$ is a small fraction, and if we represent it by e, we shall have

$$\sin M = \frac{EC}{CM} \sin AEM$$
$$= e \sin(\theta - \alpha)$$
$$= e \sin(pt + M - \alpha),$$
$$\tan M = \frac{e \sin(pt - \alpha)}{1 - e \cos(pt - \alpha)};$$

this would give M, and then θ by the formula $\theta = pt + M$

This was called an *eccentric*, and the value of e was called the *eccentricity*, which, for the moon, Hipparchus fixed at sin 5° 1′.

110 Another method of considering the motion was by means of an *epicycle*, which led to the same result

A small circle PM (fig 13), with a radius equal to EC of (fig 12), has its centre in the circumference of the circle RPD (which has the same radius as that of the eccentric), and moves round E with the uniform angular velocity p, the body M being carried in the circumference of the smaller circle, the radius PM remaining parallel to itself, or, which is the same thing, revolving from the radius PE with the same angular velocity p, so that the angle EPM equals PEA.

Now, when the angle AEP equals the angle ACM of the former figure, it is easily seen that the two triangles EPM, ECM are equal, and therefore the distance EM and the angle AEM will be the same in both, that is, the two motions are identical

111 The value of e being small, we find, rejecting e^2, &c ,

$$M = e \sin(pt - \alpha),$$

therefore
$$\theta = pt + e \sin(pt - \alpha).$$

If we reject terms of the second order in our expression for the longitude, and make $c = 1$, we get, Art (51),

$$\theta = pt + 2e \sin(pt - \alpha),$$

which will be identical with the above if we suppose the eccentricity of the eccentric to be double that of the elliptic orbit

Ptolemy (A D 140) calculated the eccentricity of the moon's orbit, and found for it the same value as Hipparchus, viz

$$\sin 5^\circ\ 1' = \tfrac{1}{12}, \text{ nearly}$$

The eccentricity in the elliptic orbit is, we know, about $\frac{1}{20}$. These values will pretty nearly reconcile the two values of θ given above, and this shews us, that for a few revolutions the moon may be considered as moving in an eccentric, and her positions in longitude calculated on this supposition will be correct to the first order.

Her distances from the earth will not however agree, for the ratio of the calculated greatest and least distances would be $\dfrac{1+\frac{1}{12}}{1-\frac{1}{12}}$ or $\frac{13}{11}$, while that of the true ones would be $\dfrac{1+\frac{1}{20}}{1-\frac{1}{20}}$ or $\frac{21}{19}$, which differ by $\frac{1}{13}$

It would, therefore, have required two different eccentrics to account for the changes in the moon's longitude and in her radius vector Changes in the latter could not, however, be easily observed with the rude instruments the ancients possessed, and it was very long before this inconsistency was detected

112 We have said that the moon's longitude, calculated on the hypothesis of an eccentric, will be pretty accurate for a *few* revolutions

The data requisite for this calculation are, the mean angular motion of the moon, the position of the apogee, and the magnitude of the eccentricity

But it was known to Hipparchus and to the astronomers of his time, that the point of the moon's orbit where she seems to move slowest, is constantly changing its position among the stars Now this point is the apogee of Hipparchus's eccentric, and he found that he could very conveniently take account of this further change by supposing the eccentric itself to have an angular motion about the earth in the same direction as the moon herself, so as to make a complete revolution in about nine years, or about 3° in each revolution *

This motion of the apsidal line follows also from our expression for the longitude, as shewn in Art (66) It is there, however, connected with an ellipse instead of an eccentric; and though the discovery that the elliptic is the true form of the fundamental orbit was not the next in the order of time after those of Hipparchus, yet, as all the irregularities which were discovered in the intervening seventeen centuries are common both to Hipparchus's eccentric and to Kepler's ellipse, it will be as well for us to consider at once this new form of the orbit

Elliptic Form of the Orbit

113. We need not dwell on the steps which led to this great and important discovery Kepler, finding that the predicted places of the planet Mars, as given by the circular theories then in use, did not always agree with the computed ones, sought to reconcile these variances by other combinations of circular orbits, and after a great number of attempts and failures, and eight years of patient investigation, he found it necessary to discard the eccentrics and epicycles altogether, and to adopt some new supposition An ellipse with the sun in the focus was at last his fortunate hypothesis, which was found to give results in accordance with observation, and this form of the orbit was, with equal success, afterwards extended to the moon as a substitute for the eccentric. but the departures from elliptic motion, due to the disturbing force of the sun, are, in the case of the moon, much greater than the disturbances of the planet Mars by the other planets

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* On the supposition of an epicycle, this motion of the apse could as easily be represented by supposing the radius which connects the moon with the centre of the epicycle to have this uniform angular velocity of about 3° in each revolution, and also in the same direction

In Kepler's hypothesis, then, the earth is to be considered as occupying the focus of an ellipse, in the perimeter of which the moon is moving, no longer with either uniform linear or angular velocity, but in such a manner that the radius vector sweeps over equal areas in equal times

This agrees with our investigation of the motion of two bodies, Art (10)

Evection

114. The hypothesis of an eccentric, whose apse line has a progressive motion, as conceived by Hipparchus, served to calculate with considerable accuracy the circumstances of eclipses, and observations of eclipses, requiring no instruments, were then the only ones which could be made with sufficient exactness to test the truth or fallacy of the supposition

Ptolemy (A D 140) having constructed an instrument, by means of which the positions of the moon could be observed in other parts of her orbit, found that they sometimes agreed, but were more frequently at variance with the calculated places, the greatest amount of error always taking place at quadrature and vanishing altogether at syzygy.

What must, however, have been a source of great perplexity to Ptolemy, when he attempted to investigate the law of this new irregularity, was to find that it did not return in every quadrature,—in some quadratures it totally disappeared, and in others amounted to 2° 39′, which was its maximum value

By dint of careful comparison of observations, he found that the value of this second inequality in quadrature was always proportional to that of the first in the same place, and was additive or subtractive according as the first was so and thus, when the first inequality in quadrature was at its maximum or 5° 1′, the second increased it to 7° 40′, which was the case when the apse line happened to be in syzygy at the same time *

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* It would seem as if Hipparchus had felt the necessity for some further modification of his first hypothesis, though he was unable to determine it, for there is an observation made by him on the moon in the position here specified

But if the apse line was in quadrature at the same time as the moon, the second inequality vanished as well as the first.

The mean value of the two inequalities combined was therefore fixed at 6° 20½′.

115 To represent this new inequality, which was subsequently called the *Evection*, Ptolemy imagined an eccentric in the circumference of which the centre of an epicycle moved while the moon moved in the circumference of the epicycle.

The centre of the eccentric and of the epicycle he supposed in syzygy at the same time, and both on the same side of the earth

Thus, if *E* represent the earth (fig 14),
S .. sun,
M moon,
c the centre of the eccentric *RKT* in syzygy,
R, the centre of the epicycle, would also be in syzygy.

Now conceive *c*, the centre of the eccentric, to describe a small circle about *E* in a retrograde direction *cc*′, while *R*, the centre of the epicycle, moves in the opposite direction, in such a manner that each of the angles *S′Ec′*, *S′ER′* may be equal to the synodical motion of the moon, that is, her mean angular motion from the sun; *SES′* being the motion of the sun in the same time.

Now we have seen, Art (110), that the first inequality was accounted for by supposing the epicycle *RM* to move into the position *rm*, *r* and *R* being at the same distance from *E*, and *rm* parallel to *RM*,* the first inequality being the angle *rEm* But when the centre of the epicycle is at *R′*, and *R′M′* is parallel to *rm*, the inequality becomes *R′EM′*, and we have a second correction or inequality *mEM′*.

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when the error of his tables would be greatest, and at a time also when she was in the nonagesimal, so that any error of longitude, arising from her yet uncertain parallax, would be avoided Ptolemy, who records the observation, employs it to calculate the evection, and obtains a result agreeing with that of his own observations (See Delambre, *Ast Ancienne*)

* For simplicity we leave out of consideration the motion of the apse

116 That this hypothesis will account for the phenomena observed by Ptolemy, Art (114), will be readily understood

At syzygies, whether conjunction or opposition, the centres of the eccentric and epicycle are in one line with the earth and on the same side of it, the points r and R' coincide, as also m and M'. Hence $mEM' = 0$

At quadratures (figs 15 and 16) c' and R' are in a straight line on opposite sides of the earth, and therefore R' and r at their furthest distance If, however, M' and m be at the same time in this line, or, in other words, if the apse line be in quadratures (fig 15), the angle mEM' will still be zero, or there will be no error in the longitude But, if the apse line is in syzygy (fig 16), the angle mEM' attains its greatest value *

Ptolemy, as we have said, found this greatest value to be 2° 39′, the angle mEr being then 5° 1′

117. Copernicus (A D 1543), having seen that Ptolemy's hypothesis gave distances totally at variance with the observations on the changes of apparent diameter,† made another and a simpler one which accounted equally well for the inequality in longitude, and was at the same time more correct in its representation of the distances

Let E (fig 17) be the earth, OD an epicycle whose centre C describes the circle $C'CC''$ about E with the moon's mean angular velocity

Let CO, a radius of this epicycle, be parallel to the apse line EA, and about O as centre let a second small epicycle be described, the radii CO and OM being so taken that

$$\frac{CO-OM}{CE} = \sin 5^\circ 1', \text{ and } \frac{CO+OM}{CE} = \sin 7^\circ 40'.$$

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* If Ptolemy had used the hypothesis of an eccentric instead of an epicycle for the first inequality of the moon, an epicycle would have represented the second inequality more simply than his method did Dr Whewell's *History of the Inductive Sciences*, vol I p 230

† See Delambre, *Ast Moderne*, vol I p 116 Whewell's *History of Inductive Sciences*, vol I p 395

The radius OM must now be made to revolve from the radius OC twice as rapidly as EC moves from ES, so that the angle COM may be always double of the angle CES.

From this construction, it follows that in syzygies the angle CES being 0° or 180°, the angle COM is 0° or 360°; and therefore C and M are at their nearest distances, as in the positions C' and C''' in the figure Then $CM = CO - OM$, and the angle CEM will range between 0° and 5° 1′, the greatest value being attained when the apse line is in quadrature

When the moon is in quadrature $CES = 90°$ or 270°, and therefore, $COM = 180°$ or 540° and C and M are at their greatest distance apart, as in the position C'', then, $CM = CO + OM$, and the angle CEM will range between 0° and 7° 40′, the former value when the apse line is itself in quadrature, and the latter when it is in syzygy

118 Thus the results attained by Ptolemy's construction are, as far as the longitudes at syzygies and quadratures are concerned, as well represented by that of Copernicus, and the variations in the distances of the moon will be far more exact, the least apparent diameter being 28′ 45″ and the greatest 37′ 33″; whereas, Ptolemy's would make the greatest diameter 1°[*]

The values which modern observations give vary between 28′ 48″ and 33′ 32″

119 It will not now be difficult to shew that the introduction of this small epicycle corresponds with that of the term $\frac{15}{4}me \sin\{(2-2m-c)\,pt - 2\beta + \alpha\}$ in our value of θ

For, referring to (fig 17), we have

$$\begin{aligned} OEM = \sin OEM &= \frac{OM}{OE}\sin OME \\ &= \frac{OM}{OE}\sin(COM - AEM) \\ &= \frac{OM}{OE}\sin(2\,SEC - AEM) \\ &= \frac{OM}{OE}\sin\{2(\text{moon's } \textit{mean} \text{ long} - \text{sun's long}) \\ &\qquad - (\text{moon's } \textit{true} \text{ long.} - \text{long of apse})\}, \end{aligned}$$

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[*] Delambre, *Ast Moderne*

and OEM being a small angle whose maximum is 1° 19½′, we may write moon's mean longitude instead of the true in the argument, and also EC for OE, therefore,

$$OEM = \frac{OM}{EC} \sin\{2(\text{moon's mean longitude} - \text{sun's longitude}) \\ - (\text{moon's mean longitude} - \text{longitude of apse})\}$$

$$= 79\tfrac{1}{2}' \sin[2\{pt - (mpt + \beta)\} - \{pt - \overline{(1-c)pt + \alpha}\}]$$

$$= 4770'' \sin\{(2 - 2m - c)pt - 2\beta + \alpha\}.$$

The value of the coefficient is from modern observations found to be 4589 61″

120 In Art (70), we have considered the effect of this second inequality in another light, not simply as a small quantity additional to the first or elliptic inequality, but as forming a part of this first; and therefore, modifying and constantly altering the eccentricity and the uniform progression of the apse line.

Boulliaud (A D. 1645), by whom the term Evection was first applied to the second inequality, seems to hint at something of this kind in the rather obscure explanations of *his* lunar hypothesis, which never having been accepted, it would be useless to give an account of.*

In Ptolemy's theory, Art (115), the evection was the result of an *apparent* increase of the first lunar epicycle caused by its approaching the earth at quadratures; but, in this second method, it is the result of an *actual* change in the elements of the elliptic orbit.

D'Arzachel, an Arabian astronomer, who observed in Spain about the year 1080, seems to have discovered the unequal motion of the apsides, but his discovery must have been lost sight of, for Horrocks, about 1640, rediscovered it 'in consequence of his attentive observations of the lunar diameter he found that when the distance of the sun from the moon's apogee was about 45° or 225°, the apogee was more advanced by 25° than when that distance was about 135° or 315° The apsides, therefore, of the moon's orbit were sometimes progressive and sometimes regressive, and required an equation of 12° 30′, sometimes additive to their mean place and sometimes subtractive from it '[*]

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* Après avoir établi les mouvemens et les époques de la lune, Boulliaud revient à l'explication de l'évection ou de la seconde inégalité. Si sa théorie n'a pas fait fortune, le nom du moins est resté 'En même temps que la lune 'avance sur son cône autour de la terre, *tout le système de la lune est déplacé ;* 'la terre emportant la lune, rejette loin d'elle l'apogée, et rapproche d'autant 'le perigee, mais cette évection à des bornes fixées'

Delambre, *Hist de l'Ast Mod*, tom ii p 157

Horrocks also made the eccentricity variable between the limits 06686 and 04362.

The combination of these two suppositions was a means of avoiding the introduction of Ptolemy's eccentric or the second epicycle of Copernicus. their joint effect constitutes the evection

Variation

121. After the discovery of the evection by Ptolemy, a period of fourteen centuries elapsed before any further addition was made to our knowledge of the moon's motions Hipparchus's hypothesis was found sufficient for eclipses, and when corrected by Ptolemy's discovery, the agreement between the calculated and observed places was found to extend also to quadratures; any slight discrepancy being attributed to errors of observation or to the imperfection of instruments

But when Tycho Brahé (A D 1580) with superior instruments extended the range of his observations to all intermediate points, he found that another inequality manifested itself Having computed the places of the moon for different parts of her orbit and compared them with observation, he perceived that she was always in advance of her computed place from syzygy to quadrature, and behind it from quadrature to syzygy, the maximum of this *variation* taking place in the octants, that is, in the points equally distant from syzygy and quadrature The moon's velocity therefore, so far as this inequality was concerned, was greatest at new and full moon, and least at the first and third quarter *

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[*] Small's *Astronomical Discoveries of Kepler*, p 307

Tycho fixed the maximum of this inequality at 40′ 30″ The value which results from modern observations is 39′ 30″

122 We have already two epicycles, or one epicycle and an eccentric, to explain the first two inequalities· by the introduction of another epicycle or eccentric, the variation also might have been brought into the system, but Tycho adopted a different method † like Ptolemy, he employed an eccentric for the evection, but for the first or elliptic inequality he employed a couple of epicycles, and this complicated combination, which it is needless further to describe, represented the change of distance better than Ptolemy's

To introduce the *variation*, he imagined the centre of the larger epicycle to librate backwards and forwards on the eccentric, to an extent of $40\frac{1}{2}'$ on each side of its mean position, this mean place itself advancing uniformly along the eccentric with the moon's mean motion in anomaly, and the libration was so adjusted, that the moon was in her mean place at syzygy and quadrature, and at her furthest distance from it in the octants, the period of a complete libration being half a synodical revolution

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* 'It appears that Mohammed-Aboul Wefa al-Bouzdjani, an Arabian astro-'nomer of the tenth century, who resided at Cairo, and observed at Bagdad in '975, discovered a third inequality of the moon, in addition to the two ex-'pounded by Ptolemy, the equation of the centre and the evection This 'third inequality, the variation, is usually supposed to have been discovered 'by Tycho Brahe, six centuries later In an almagest of Aboul-Wefa, a 'part of which exists in the Royal Library at Paris, after describing the two 'inequalities of the moon, he has a Section IX, "Of the third anomaly of the 'moon called *Muhazal* or *Prosneusis*" But this discovery of Aboul-Wefa 'appears to have excited no notice among his contemporaries and followers, at 'least it had been long quite forgotten, when Tycho Brahé rediscovered the 'same lunar inequality' Whewell's *Hist of Inductive Sciences*, vol i p 243.

† For a full description of Tycho's hypothesis, see Delambre, *Hist de l'Ast. Mod*, tom i p 162, and *An Account of the Astronomical Discoveries of Kepler*, by Robert Small, p 135

Annual Equation

123. Tycho Brahé was also the discoverer of the fourth inequality, called the annual equation. This was connected with the anomalistic motion of the sun, and did not, like the previous inequalities, depend on the position of the moon in her orbit.

Having calculated the position of the moon corresponding to any given time, he found that the observed place was behind her computed one while the sun moved from perigee to apogee, and before it in the other half year.

Tycho did not state this distinctly, but he made a correction which, though wrong in quantity and applied in an indirect manner, showed that he had seen the necessity and understood the law of this inequality.

He did not try to represent it by any new eccentric or epicycle, but he increased by (8m. 13s.) sin(*sun's anomaly*) the time which had served to calculate the moon's place,* thus assuming that the true place, after that interval, would agree with the calculated one. Now, as the moon moves through 4′ 30″ in 8m. 13s., it is clear that adding (8m. 13s.) sin(*sun's anomaly*) to the time is the same thing as subtracting (4′ 30″) sin(*sun's anomaly*) from the calculated longitude, which was therefore the correction virtually introduced by Tycho.† Modern observations shew the coefficient to be 11′ 9″.

We have seen, Art. (75), how this inequality may be inferred from our equations.

Reduction

124. The next inequality in longitude which we have to consider, is not an inequality in the same sense as the foregoing,

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* That is, the equation of time which he used for the moon differed by that quantity from that used for the sun.

† Horrocks (1639) made the correction in the same manner as Tycho, but so increased it that the corresponding coefficient was 11′ 51″ instead of 4′ 30″. Flamsteed was the first to apply the correction to the longitude instead of the time.

that is, it does not arise from any irregularity in the motion of the moon herself in her orbit, but simply because that orbit is not in the same plane as that in which the longitudes are reckoned, so that even a regular motion in the one would be necessarily irregular when referred to the other Thus if NMn (fig 10) be the moon's orbit and ΥNm the ecliptic, and if M the moon be referred to the ecliptic by the great circle Mm perpendicular to it, then MN and mN are 0°, 90°, 180°, 270°, and 360° simultaneously, but they differ for all intermediate values the difference between them is called the *reduction*

The difference between the longitude of the node and that of the moon in her orbit being known, that is the side NM of the right-angled spherical triangle NMm, and also the angle N the inclination of the two orbits, the side Nm may be calculated by the rules of spherical trigonometry, and the difference between it and NM, applied with a proper sign to the longitude in the orbit, gives the longitude in the ecliptic

Tycho was the first to make a table of the reduction instead of calculating the spherical triangle His formula was

$$\text{reduction} = \tan^2\tfrac{1}{2}I \sin 2L - \tfrac{1}{2}\tan^4\tfrac{1}{2}I \sin 4L,$$

where I is the inclination of the orbit and L the longitude of the moon diminished by that of the node

The first term corresponds with the term $-\frac{1}{4}k^2 \sin 2(gpt-\gamma)$ of the expression for θ

Latitude of the Moon

125 That the moon's orbit is inclined to the ecliptic was known to the earliest astronomers, from the the non-recurrence of eclipses at every new and full moon; and it was also known, since the eclipses did not always take place in the same part of the heavens, that the line of nodes represented by Nn (fig 10) has a retrograde motion on the ecliptic, N moving towards Υ

Hipparchus fixed the inclination of the moon's orbit to the ecliptic at 5°, which value he obtained by observing the greatest distance at which she passes to the north or south of some star known to be in or very near the ecliptic, as for instance the bright star Regulus; and by comparing the recorded eclipses from the times of the Chaldean astronomers down to his own, he found that the line of nodes goes round the ecliptic in a retrograde direction in about $18\frac{2}{3}$ years

This result is indicated in our expression for the value of the latitude by the term $k\sin(g\theta-\gamma)$, as we have shewn Art (78)

126 Tycho Brahé further discovered that the inclination of the lunar orbit to the ecliptic was not a constant quantity of 5° as Hipparchus had supposed, but that it had a mean value of 5° 8′, and ranged through 9′ 30″ on each side of this, the least inclination 4° $58\frac{1}{2}'$ occurring when the node was in syzygy, and the greatest 5° $17\frac{1}{2}'$ being attained when the node was in quadrature *

He also found that the retrograde motion of the node was not uniform · the mean and true position of the node agreed very well when they were in syzygy or quadrature, but they were 1° 46′ apart in the octants

By referring to Art (80), we shall see that these corrections, introduced by Tycho Brahé, correspond to the second term of our expression for *s*

Since Hipparchus could observe the moon with accuracy only in the eclipses, at which time the node is in or near syzygy,

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* Ebn Jounis, an Arabian astronomer (died A D 1008), whose works were translated about 30 years since by Mons Sedillot, states that the inclination of the moon's orbit had been often observed by Aboul Hassan Aly-ben-Amajour about the year 918, and that the results he had obtained were generally greater than the 5° of Hipparchus, but that they *varied considerably*

Ebn Jounis adds, however, that he himself had observed the inclination several times and found it 5° 3′, which leads us to infer that he always observed in similar circumstances, for otherwise a variation of nearly 23′ could scarcely have escaped him See Delambre, *Hist de l'Ast du Moyen Age*, p. 139

The mean value of the inclination is 5° 8′ 55 46″,—the extreme values are 4° 57′ 22′ and 5° 20′ 6″

The mean daily motion of the line of nodes is 3′ 10 64″, or one revolution in 6793 39 days, or 18y 218d 21h 22m 46s we see why the value he found for the inclination of the orbit was approximately its minimum value, and also why he was unable to detect the want of uniformity in the motion of the node.

127 To represent these changes in the position of the moon's orbit, Tycho made the following hypothesis.

Let *ENF* (fig 18) be the ecliptic, *K* its pole, *BAC* a small circle, having also *K* for pole and at a distance from it equal to 5° 8′. Then, if we suppose the pole of the moon's orbit to move uniformly in the small circle and in the direction *BAC*, the node *N*, which is at 90° from both *A* and *K*, will retrograde uniformly on the ecliptic, and the inclination of the two orbits will be constant and equal to *AK*

But instead of supposing the pole of the moon's orbit to be at *A*, let a small circle *abcd* be described with *A* as pole and a radius of 9′ 30″, and suppose the pole of the moon's orbit to describe this small circle with double the velocity of the node in its synodical revolution which is accomplished in about 346 days, in such a manner that when the node is in syzygy the pole may be at *a*, the nearest point to *K*, and at *c* the most distant point when the node comes to quadrature, at *b* in the first and third octants, and at *d* in the second and fourth, so as to describe the small circle in about 173 days, the centre *A* of the small circle retrograding meanwhile with its uniform motion

By this method of representing the motion, we see that

$$\text{when node is in }\left.\begin{array}{r}\text{syzygy}\\ \text{quadrature}\end{array}\right\}\text{ the inclination of the orbit is }\left\{\begin{array}{l}Ka = 5^\circ\, 8' - 9\tfrac{1}{2}' = 4^\circ\, 58\tfrac{1}{2}',\\ Kc = 5^\circ\, 8' + 9\tfrac{1}{2}' = 5^\circ\, 17\tfrac{1}{2}',\end{array}\right.$$

while at the octants it has its mean value $Kb = Kd = KA = 5^\circ\, 8'$

Again, with respect to the motion of the node, since *N* is the pole of *KaAc*, it follows that when in syzygy and quadrature, the node occupied its mean place, in the first and third octants, the pole being at *b*, the node was behind its mean place by the angle $bKA = (9'\, 30'')$ cosec $5^\circ\, 8' = 1^\circ\, 46'$, nearly, and it was as much in advance of its true place in the second and fourth octants

So that the whole motion of the node, and the correction which Tycho had discovered, were properly represented by this hypothesis, which is exactly similar to that which Copernicus had imagined to explain the precession of the Equinox.

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fore, hereafter omit such terms as $A\cos(J - B)$ altogether, and merely write

$$u = a\{1 + e\cos(c\theta - \alpha)\} +$$

38 So far, all that we know about c is that it differs from unity at most by a quantity of the first order, but its value will be more and more correctly obtained by always writing, in the successive approximations, $a + ae\cos(c\theta - \alpha)$ for the first two terms of the value of u, then the coefficient of $\cos(c\theta - \alpha)$ in the differential equation must equal $ae(1 - c^2)$; and this will enable us to determine c to the same order of approximation as that of the differential equation itself See Arts (48) and (94)

39 In carrying on the solution of s, the same difficulty arises as in u, and it will be found necessary to change it into

$$s = k\sin(g\theta - \gamma) \quad . \quad . \qquad S'_1,$$

g being a quantity which differs from unity at most by a quantity of the first order See Arts (49) and (95)

40 The equation Θ_1 will also be modified by this change in the value of u,

$$\frac{dt}{d\theta} = \frac{1}{ha^2}\,\frac{1}{\{1 + e\cos(c\theta - \alpha)\}^2};$$

$$\text{or } \frac{d\,ct}{d\,c\theta} = \frac{1}{ha^2}\,\frac{1}{\{1 + e\cos(c\theta - \alpha)\}^2}$$

Here $c\theta$ and ct hold the places which θ and t occupied in (33), therefore

$$c\theta = cpt + 2e\sin(cpt - \alpha),$$

$$\text{or} \quad \theta = pt + 2e\sin(cpt - \alpha). \qquad \Theta'_1$$

to the first order, since $\frac{e}{c} = e$ to the first order

41. Since the disturbing forces are to be taken into account in the next approximation, we shall have to use the value of u' found in (18), which is

$$u' = a'\{1 + e'\cos(\theta' - \zeta)\}$$

but this introduces θ', we must therefore further modify it by substituting for θ' its value in terms of θ, and it will be found sufficient, for the purpose of the present work, to obtain the connexion between them to the first order, which may be done as follows

Let m be the ratio of the mean motions of the sun and moon,

p', p their mean angular velocities, $p' = mp$,

$p't + \beta$, pt mean longitudes at time t, β being the sun's longitude when $t = 0$,

θ', θ true longitudes at time t,

ζ, α longitude of perigees when $t = 0$,

therefore $\theta' - \zeta =$ sun's true anomaly,

and $p't + \beta - \zeta =$ mean anomaly

But, by Art (13),

true anomaly = mean anomaly + $2e'$ sin (mean anomaly) + &c ,

$$\begin{aligned} \text{therefore}\quad \theta' &= p't + \beta + 2e' \sin(p't + \beta - \zeta) + \\ &= mpt + \beta + 2e' \sin(mpt + \beta - \zeta) + \\ &= m\theta + \beta + 2e' \sin(m\theta + \beta - \zeta) \end{aligned}$$

to the first order,

because $pt = \theta - 2e \sin(c\theta - \alpha)$ to the first order by (40)

Whence $u' = a' \{1 + e' \cos(m\theta + \beta - \zeta)\}$ to the first order

42 The values of $\sin 2(\theta - \theta')$ and $\cos 2(\theta - \theta')$ can also be readily obtained to the same order

$$\begin{aligned} &\sin 2(\theta - \theta') \\ &= \sin\{(2 - 2m)\,\theta - 2\beta - 4e' \sin(m\theta + \beta - \zeta)\} \\ &= \sin\{(2 - 2m)\,\theta - 2\beta\} - 4e' \sin(m\theta + \beta - \zeta) \cos\{(2 - 2m)\,\theta - 2\beta\} \\ &= \sin\{(2 - 2m)\,\theta - 2\beta\} - 2e' \sin\{(2 - m)\,\theta - \beta - \zeta\} \\ &\qquad + 2e' \sin\{(2 - 3m)\,\theta - 3\beta + \zeta\} \end{aligned}$$

Similarly, $\cos 2(\theta - \theta') = \cos\{(2 - 2m)\,\theta - 2\beta\}$

$$- 2e' \cos\{(2 - m)\,\theta - \beta - \zeta\} + 2e' \cos\{(2 - 3m)\,\theta - 3\beta + \zeta\}$$

The first term of each of these is all we shall require

SECTION III

To solve the Equations to the Second Order

43 Let us recapitulate the results of the last approximation

$$u = a\{1 + e\cos(c\theta - \alpha)\},$$
$$u' = a'\{1 + e'\cos(m\theta + \beta - \zeta)\},$$
$$s = k\sin(g\theta - \gamma),$$
$$\theta - \theta' = (1 - m)\,\theta - \beta - 2e'\sin(m\theta + \beta - \zeta)$$

These values must now be substituted in the expressions for $\frac{P}{h^2u^2}$, $\frac{T}{h^2u^3}$, $\frac{Ps - S}{h^2u^3}$, $\frac{T}{h^2u^3}\frac{du}{d\theta}$, $\frac{T}{h^2u^3}\frac{ds}{d\theta}$, $\left(\frac{d^2u}{d\theta^2} + u\right)\int\frac{T}{h^2u^3}\,d\theta$, $\left(\frac{d^2s}{d\theta^2} + s\right)\int\frac{T}{h^2u^3}\,d\theta$, retaining terms above the second order, when, according to the criterion of Art (29), they promise to become of the second order after integrating

The equations (β') and (γ') of Art (30) will then assume the forms

$$\frac{d^2u}{d\theta^2} + u = F(\theta),$$

$$\frac{d^2s}{d\theta^2} + s = f(\theta),$$

and the integration of these will enable us to obtain u and s to the second order, after which, equation (α) of Art. (20) will give the connexion between θ and t to the same order.

44 The quantity $\frac{m'a'^3}{h^2a^3}$, which we shall meet with as a coefficient of the terms due to the disturbing force, can be replaced by m^2a, m being the ratio of the mean motions of the sun and moon; for, as in Art (24),

$$\frac{m'a'^3}{h^2a^3} = \frac{m'a'^3}{\mu a^2} = \frac{m'a'^3}{\mu a^3}\,a = \frac{(\text{periodic time of moon about earth})^2}{(\text{periodic time of sun about earth})^2}\,a,$$

nearly,

and $m = \dfrac{p'}{p} = \dfrac{\text{periodic time of moon about earth}}{\text{periodic time of sun about earth}}$, nearly,

therefore $\dfrac{m'a'^3}{h^2a^3}$ may be replaced by m^2a *

45 We have therefore

$$\frac{P}{h^2u^3} = \frac{\mu}{h^2}\left(1 - \tfrac{3}{2}s^2\right) - \frac{m'u'^3}{h^2u^3}\left\{\tfrac{1}{2} + \tfrac{3}{2}\cos 2(\theta - \theta')\right\}$$

$$= a\left\{1 - \tfrac{3}{2}k^2\sin^2(g\theta - \gamma)\right\} - \left[\frac{m'a'^3\{1 + e'\cos(m\theta + \beta - \zeta)\}^3}{h^2a^3\{1 + e\cos(c\theta - \alpha)\}^3}\right]$$
$$\left[\tfrac{1}{2} + \tfrac{3}{2}\cos\{(2 - 2m)\theta - 2\beta\}\right]$$

$$= a\begin{cases} 1 - \tfrac{3}{4}k^2 + \tfrac{3}{4}k^2\cos 2(g\theta - \gamma) \\ - \tfrac{1}{2}m^2\{1 + 3e'\cos(m\theta + \beta - \zeta)\}\{1 - 3e\cos(c\theta - \alpha)\} \\ \qquad [1 + 3\cos\{(2 - 2m)\theta - 2\beta\}] \end{cases}$$

$$= a\begin{cases} 1 - \tfrac{3}{4}k^2 + \tfrac{3}{4}k^2\cos 2(g\theta - \gamma) - \tfrac{1}{2}m^2[1 + 3\cos\{(2 - 2m)\theta - 2\beta\}] \\ - \tfrac{3}{2}m^2e'\cos(m\theta + \beta - \zeta) + \tfrac{3}{2}m^2e\cos(c\theta - \alpha) \\ \qquad + \tfrac{9}{4}m^2e\cos\{(2 - 2m - c)\theta - 2\beta + \alpha\} \end{cases}$$

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* It may perhaps be imagined, since the orbits described are not accurately ellipses, and, even if they were, a' and a would be the reciprocals of the semi-latera recta, and not those of the semi major axes, that therefore $\frac{m'a'^3}{h^2a^3}$ is only approximately equal to m^2a, and that if it were required to carry our investigations to a higher order than we propose to do, it would be necessary to correct this equation

But the very fact that the moon's orbit is not an exact ellipse, shews that the yet unknown quantity a is not a definite magnitude whose value is fixed at the outset like those of m' and μ It is a constant, it is true, but it is one whose value may be whatever we please, provided the assumption do not interfere with our continuous approximation

Now from Art (30), where a was first introduced, we see that it is equal to $\frac{\mu}{h^2}$, h^2 being itself an *arbitrary* constant brought in by the integration, and this being the only relation between the quantities h and a, we are at liberty to establish any second one The relation $\frac{m'a'^3}{h^2a^3} = m^2a$ is this second assumed relation The reasoning in the text does not *prove* this relation, since, as we say, it is an arbitrary one, but it suggests it The actual value of a is now fixed, and the method of obtaining it from observation will be explained in the

The last three terms are retained, though of the third order, according to Art (29) · The first of the three will not rise in importance in the value of u, but it is retained for its subsequent use in finding t, when it will become of the second order The other terms of the third order, which arise in the development of the expression, are neglected, as the coefficients of θ in their arguments are neither small nor near unity

$$\begin{aligned}\frac{T}{h^2u^3} &= -\frac{3m'u'^3}{2h^2u^4}\sin 2(\theta-\theta')\\
&= -\frac{3m'a'^3\{1+e'\cos(m\theta+\beta-\zeta)\}^3}{2h^2a^4\{1+e\cos(c\theta-\alpha)\}^4}\sin\{(2-2m)\,\theta-2\beta\}\\
&= -\tfrac{3}{2}m^2\,\{1+3e'\cos(m\theta+\beta-\zeta)\}\\
&\qquad\{1-4e\cos(c\theta-\alpha)+10e^2\cos^2(c\theta-\alpha)\}\sin\{(2-2m)\,\theta-2\beta\}\\
&= -\tfrac{3}{2}m^2\,\{1+3e'\cos(m\theta+\beta-\zeta)\}\\
&\qquad[\sin\{(2-2m)\,\theta-2\beta\}-2e\sin\{(2-2m-c)\,\theta-2\beta+\alpha\}\\
&\qquad\qquad+\tfrac{5}{2}e^2\sin\{(2-2m-2c)\,\theta-2\beta+2\alpha\}]\\
&= -\tfrac{3}{2}m^2\begin{cases}\sin\{(2-2m)\,\theta-2\beta\}-2e\sin\{(2-2m-c)\,\theta-2\beta+\alpha\}\\ +\tfrac{5}{2}e^2\sin\{(2-2m-2c)\,\theta-2\beta+2\alpha\}\end{cases}.\end{aligned}$$

We have, in the course of the reduction, dropped those terms which, according to Art (29), could not produce important terms in the resulting value either of u or of t The last term, though of the fourth order, is retained because $2-2m-2c$ is small

$$\begin{aligned}\frac{T}{h^2u^3}\,\frac{du}{d\theta} &= (\text{previous expression})\ \{-aec\sin(c\theta-\alpha)\}\\
&= \tfrac{3}{4}m^2ae\cos\{(2-2m-c)\,\theta-2\beta+\alpha\},\end{aligned}$$

for to the first order $c=1$;

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next chapter If we had assumed $\frac{m'a'^3}{h^2a^3}=m^2\lambda a$, where λ is any *given* quantity differing very slightly from unity, we could still have proceeded with our approximations, for the corresponding value of a, though slightly different from the former, would still, when used in the equation U_1', give a value of u approximate to the first order

But if λ differed considerably from unity, the value of a which this equation would furnish, would no longer render U_1' an approximation to the first order, and if we made use of the equation U_1', we should have no guarantee that the next step would be a closer approximation

$$\frac{T}{h^2u^3}\frac{ds}{d\theta} = (\text{same expression})\ \{kg\cos(g\theta-\gamma)\}$$
$$= -\tfrac{3}{4}m^2h\sin\{(2-2m-g)\,\theta-2\beta+\gamma\},$$
for $g = 1$ to the first order

$$\int\frac{T}{h^2u^3}d\theta = \tfrac{3}{2}m^2\left\{\begin{array}{l}\dfrac{1}{2-2m}\cos\{(2-2m)\,\theta-2\beta\} \\ -\dfrac{2e}{2-2m-c}\cos\{(2-2m-c)\,\theta-2\beta+\alpha\} \\ +\dfrac{5e^2}{2(2-2m-2c)}\cos\{(2-2m-2c)\,\theta-2\beta+2\alpha\}.\end{array}\right.$$

but $$\frac{1}{2-2m} = \tfrac{1}{2} + \text{terms of first order},$$
$$-\frac{2e}{2-2m-c} = -2e + \text{terms of second order},$$
$$\frac{5e^2}{2(2-2m-2c)} = \frac{5e^2}{4(1-m-c)}$$

Here the denominator is of the first order, and cannot be further simplified without a more accurate knowledge of the value of c We shall find in the next value of u, Art (48), that $1-c$ is of the second order, and as this result is obtained independently of the term we are here considering, which is only retained for the sake of finding t, there is no impropriety in anticipating thus far in order to simplify this coefficient, which then becomes

$$-\frac{5e^2}{4m} + \text{terms of the second order},$$

$$\therefore \int\frac{T}{h^2u^3}d\theta = \tfrac{3}{4}m^2\cos\{(2-2m)\,\theta-2\beta\}$$
$$-3m^2e\cos\{(2-2m-c)\,\theta-2\beta+\alpha\}$$
$$-\tfrac{15}{8}me^2\cos\{(2-2m-2c)\,\theta-2\beta+2\alpha\}.$$

Also, by Art (36),

$$\frac{d^2u}{d\theta^2} + u = a + \text{quantities of the second order},$$
$$\frac{d^2s}{d\theta^2} + s = \text{small quantity of the second order at least},$$

$$2\left(\frac{d^2u}{d\theta^2}+u\right)\int\frac{T}{h^2u^3}\,d\theta = \tfrac{3}{2}m^2a\cos\{(2-2m)\,\theta-2\beta\}$$
$$-\,6m^2ae\cos\{(2-2m-c)\,\theta-2\beta+\alpha\}$$
$$-\,\tfrac{15}{4}me^2a\cos\{(2-2m-2c)\,\theta-2\beta+2\alpha\},$$

$$2\left(\frac{d^2s}{d\theta^2}+s\right)\int\frac{T}{h^2u^3}\,d\theta = 0, \text{ to the third order}$$

Lastly,

$$\frac{Ps-S}{h^2u^3} = -\,\frac{m'su'^3}{h^2u^4}\,\{\tfrac{3}{2}+\tfrac{3}{2}\cos 2(\theta-\theta')\}$$
$$= -\,\tfrac{3}{2}m^2s\,[1+\cos\{(2-2m)\,\theta-2\beta\}]$$
$$\{1+3e'\cos(m\theta+\beta-\zeta)-4e\cos(c\theta-\alpha)\}$$
$$= -\,\tfrac{3}{2}m^2k\,[\sin(g\theta-\gamma)-\tfrac{1}{2}\sin\{(2-2m-g)\,\theta-2\beta+\gamma\}]$$

In all these expressions we have rejected those terms of the third and higher orders which, according to Art (29), would not influence the second order [*]

46 We must now substitute these values in the differential equations for u and s, and then integrate, omitting the complementary term $A\cos(\theta-B)$, for though, by the theory of differential equations, this would form a necessary part of the solution, it cannot, as we have seen Art (35), be a part of the correct values of u or s

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[*] Instead of the forces which really act on the moon, we originally substituted three equivalent ones, P, T, S, these again are, by the preceding expressions, replaced by a set of others For, we may conceive each of the terms in $\frac{P}{h^2u^3}$, &c to correspond to a force,—a component of P, T, or S, each force having the same argument as the term to which it corresponds, and therefore going through its cycle of values in the same time Now, by Art (20), when the coefficient of θ in the argument is near unity, the term becomes important in the radius vector, and when near zero, in the longitude hence, a force whose period is nearly the same as that of the moon, produces important effects in the radius vector, and a force whose period is very long will be important in its effect on the longitude

See *Airy's* Tracts, *Planetary Theory*, p 78

47 Since the form of the solution is known, the actual expressions for u and s will be obtained with more facility by assuming them with arbitrary coefficients, the values of which are afterwards determined by substitution

We must remember, however, that the coefficients of $\cos(c\theta-\alpha)$ in u and of $\sin(g\theta-\gamma)$ in s must be assumed the same as in the first approximate solutions, and that these assumptions will enable us to obtain the values of c and g to the same order of approximation as that to which we are working, Art (38)

48 Considering, firstly, the equation in u, we have

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} - \frac{T}{h^2u^3}\frac{du}{d\theta} - 2\left(\frac{d^2u}{d\theta^2} + u\right)\int\frac{T}{h^2u^3}\,d\theta,$$

$$= a\left\{\begin{aligned} &1 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2 \\ &+ \tfrac{3}{2}m^2e\cos(c\theta-\alpha) \\ &+ \tfrac{3}{4}k^2\cos 2(g\theta-\gamma) \\ &- 3m^2\cos\{(2-2m)\theta-2\beta\} \\ &+ \tfrac{15}{2}m^2e\cos\{(2-2m-c)\theta-2\beta+\alpha\} \\ &- \tfrac{3}{2}m^2e'\cos(m\theta+\beta-\zeta) \\ &+ \tfrac{15}{4}me^2\cos\{(2-2m-2c)\theta-2\beta+2\alpha\} \end{aligned}\right.$$

The last two terms would not be retained if we wished to find the value of u only, but, in finding t afterwards, they will become of the second order

$$\text{Assume } u = a\left\{\begin{aligned} &1 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2 \\ &+ e\cos(c\theta-\alpha) \\ &+ A\cos 2(g\theta-\gamma) \\ &+ B\cos\{(2-2m)\theta-2\beta\} \\ &+ C\cos\{(2-2m-c)\theta-2\beta+\alpha\} \\ &+ D\cos(m\theta+\beta-\zeta) \\ &+ E\cos\{(2-2m-2c)\theta-2\beta+2\alpha\}. \end{aligned}\right.$$

Then, by substitution,

$$\begin{aligned}
e\,(1-c^2) &= \tfrac{3}{2}m^2 e,\\
A\,(1-4g^2) &= \tfrac{3}{4}k^2,\\
B\,\{1-(2-2m)^2\} &= -\,3m^2,\\
C\,\{1-(2-2m-c)^2\} &= \tfrac{15}{2}m^2 e,\\
D\,(1-m^2) &= -\,\tfrac{3}{2}m^2 e',\\
E\,\{1-(2-2m-2c)^2\} &= \tfrac{15}{4}me^2;
\end{aligned}$$

whence $c = \sqrt{(1-\tfrac{3}{2}m^2)} = 1-\tfrac{3}{4}m^2,$

$$\begin{aligned}
A &= \frac{3k^2}{4\,(1-4)} &&= -\,\tfrac{1}{4}k^2,\\
B &= \frac{-\,3m^2}{1-4} &&= m^2,\\
C &= \frac{15m^2e}{2\,\{1-(1-2m+\tfrac{3}{4}m^2)^2\}} &&= \tfrac{15}{8}me,\\
D &= -\,\frac{3m^2e'}{2\,(1-m^2)} &&= -\,\tfrac{3}{2}m^2e',\\
E &= \frac{15me^2}{4\,\{1-(-2m+\tfrac{3}{2}m^2)^2\}} &&= \tfrac{15}{4}me^2
\end{aligned}$$

Therefore

$$u = a\left\{\begin{aligned}
&1-\tfrac{3}{4}k^2-\tfrac{1}{2}m^2\\
&+e\cos(c\theta-\alpha)\\
&-\tfrac{1}{4}k^2\cos 2(g\theta-\gamma)\\
&+m^2\cos\{(2-2m)\,\theta-2\beta\}\\
&+\tfrac{15}{8}me\cos\{(2-2m-c)\,\theta-2\beta+\alpha\}\\
&-\tfrac{3}{2}m^2e'\cos(m\theta+\beta-\zeta)\\
&+\tfrac{15}{4}me^2\cos\{(2-2m-2c)\,\theta-2\beta+2\alpha\}
\end{aligned}\right\}.\,U_2$$

49 The differential equation of the latitude

$$\frac{d^2s}{d\theta^2}+s=\frac{Ps-S}{h^2u^3}-\frac{T}{h^2u^3}\frac{ds}{d\theta}-2\left(\frac{d^2s}{d\theta^2}+s\right)\int\frac{T}{h^2u^3}\,d\theta$$

becomes, after substitution,

$$\frac{d^2s}{d\theta^2}+s=\begin{cases}-\frac{3}{2}m^2k\sin(g\theta-\gamma)\\+\frac{3}{2}m^2k\sin\{(2-2m-g)\,\theta-2\beta+\gamma\}\end{cases}$$

Assume $s=\begin{cases}k\sin(g\theta-\gamma)\\+A\sin\{(2-2m-g)\,\theta-2\beta+\gamma\}.\end{cases}$

Then, by substitution, we get

$$k(1-g^2)\qquad\qquad=-\tfrac{3}{2}m^2k,$$

$$A\{1-(2-2m-g)^2\}=+\tfrac{3}{2}m^2k$$

Therefore $g=\sqrt{(1+\frac{3}{2}m^2)}\qquad\qquad=1+\frac{3}{4}m^2,$

$$A=\frac{3m^2k}{2\{1-(1-2m-\frac{3}{4}m^2)^2\}}=\tfrac{3}{8}mk;$$

therefore $s=\left\{\begin{matrix}k\sin(g\theta-\gamma)\\+\frac{3}{8}mk\sin\{(2-2m-g)\,\theta-2\beta+\gamma\}\end{matrix}\right\}\qquad S_2.$

50 We can now find the connexion between the longitude and the time to the second order,

$$\frac{dt}{d\theta}=\frac{1}{hu^2\left(1+2\int\frac{T}{h^2u^3}\,d\theta\right)^{\frac{1}{2}}}=\frac{1}{hu^2}\left(1-\int\frac{T}{h^2u^3}\,d\theta\right),$$

and from Art (48) we have

$$\frac{1}{hu^2}=\frac{1}{ha^2\{1+e\cos(c\theta-\alpha)+\Sigma_2+\Sigma_3\}^2},$$

(Σ_2 being the sum of all the terms of the second, and Σ_3 those of the third order in u,)

$$=\frac{1}{ha^2}[1-2\{e\cos(c\theta-\alpha)+\Sigma_2+\Sigma_3\}$$
$$+3\{e\cos(c\theta-\alpha)+\Sigma_2+\Sigma_3\}^2-\&c.]$$

$$=\frac{1}{ha^2}[1-2e\cos(c\theta-\alpha)-2\Sigma_2-2\Sigma_3$$
$$+3e^2\cos^2(c\theta-\alpha)+6\Sigma_2e\cos(c\theta-\alpha)]$$

$$= \frac{1}{ha^2}\left[1 - 2e\cos(c\theta - \alpha) - 2\Sigma_2 - 2\Sigma_3 + \tfrac{3}{2}e^2 + \tfrac{3}{2}e^2\cos 2(c\theta - \alpha) \right.$$
$$\left. + \tfrac{45}{8}me^2\cos\{(2 - 2m - 2c)\,\theta - 2\beta + 2\alpha\}\right]$$

$$= \frac{1}{ha^2}\left\{\begin{aligned} 1 + \tfrac{3}{2}e^2 + \tfrac{3}{2}k^2 + m^2 &- 2e\cos(c\theta - \alpha)\\ &+ \tfrac{3}{2}e^2\cos 2(c\theta - \alpha)\\ &+ \tfrac{1}{2}k^2\cos 2(g\theta - \gamma)\\ &- 2m^2\cos\{(2 - 2m)\,\theta - 2\beta\}\\ &- \tfrac{15}{4}me\cos\{(2 - 2m - c)\,\theta - 2\beta + \alpha\}\\ &+ 3m^2e'\cos(m\theta + \beta - \zeta)\\ &- \tfrac{15}{8}me^2\cos\{(2 - 2m - 2c)\,\theta - 2\beta + 2\alpha\}\end{aligned}\right.$$

Also, from Art (45),

$$1 - \int \frac{T}{h^2u^3}\,d\theta = 1 - \tfrac{3}{4}m^2\cos\{(2 - 2m)\,\theta - 2\beta\}$$
$$+ \tfrac{15}{8}me^2\cos\{(2 - 2m - 2c)\,\theta - 2\beta + 2\alpha\},$$

neglecting the other term of the third order, the coefficient of the argument not being small

We have now to multiply these results together, and we see that the term having for argument $(2 - 2m - 2c)\,\theta - 2\beta + 2\alpha$ will disappear in the product If we trace this term, we shall find that it arose in $\int \frac{T}{h^2u^3}\,d\theta$, from retaining originally terms of the fourth order, but in $\frac{1}{hu^2}$ it arises from combining terms originally of the first and third orders If, therefore, we had rejected terms beyond the third order indiscriminately, the expression $\frac{dt}{d\theta}$ would have contained this term, introduced by $\frac{1}{hu^2}$, to the third order, and in t it would have been raised to the second order, and therefore formed an important part of its value instead of disappearing altogether from the expression Hence the necessity for retaining such terms of the fourth order in $\frac{T}{h^2u^3}$

$$\frac{dt}{d\theta}=\frac{1}{ha^2}\left\{\begin{aligned}1+\tfrac{3}{2}e^2+\tfrac{3}{2}k^2+m^2-2e\cos(c\theta-\alpha)&\\ +\tfrac{3}{2}e^2\cos 2(c\theta-\alpha)&\\ +\tfrac{1}{2}k^2\cos 2(g\theta-\gamma)&\\ -\tfrac{11}{4}m^2\cos\{(2-2m)\theta-2\beta\}&\\ -\tfrac{15}{4}me\cos\{(2-2m-c)\theta-2\beta+\alpha\}&\\ +3m^2e'\cos(m\theta+\beta-\zeta)&\end{aligned}\right.$$

Let $$\frac{1}{ha^2}\left(1+\tfrac{3}{2}e^2+\tfrac{3}{2}k^2+m^2\right)=\frac{1}{p},$$

therefore $$\frac{p}{ha^2}=1-\tfrac{3}{2}e^2-\tfrac{3}{2}k^2-m^2 \text{ to the third order},$$

therefore, multiplying by p and integrating, we get, still to the second order,

$$\left.\begin{aligned}pt=\theta-2e\sin(c\theta-\alpha)+\tfrac{3}{4}e^2\sin 2(c\theta-\alpha)&\\ +\tfrac{1}{4}k^2\sin 2(g\theta-\gamma)&\\ -\tfrac{11}{8}m^2\sin\{(2-2m)\theta-2\beta\}&\\ -\tfrac{15}{4}me\sin\{(2-2m-c)\theta-2\beta+\alpha\}&\\ +3me'\sin(m\theta+\beta-\zeta)&\end{aligned}\right\}\quad\Theta_2,$$

no constant is added, the time being reckoned from the instant when the mean value of θ vanishes, for the reasons explained in Art (34)

51 The preceding equations U_2, S_2, Θ_2 give the reciprocal of the radius vector, the latitude and the time in terms of the true longitude, but the principal object of the analytical investigations of the Lunar Theory being the formation of tables which give the coordinates of the moon at stated times, we must express u, s, and θ in terms of t.

To do this, we must reverse the series $pt=\theta-$ &c , and then substitute the value of θ in the expressions for u and s.

Now $$\theta=pt+2e\sin(c\theta-\alpha) \text{ to the first order}$$

$$=pt+2e\sin(cpt-\alpha)\quad .\qquad ..\quad ;$$

therefore $c\theta - \alpha = cpt - \alpha + 2e\sin(cpt - \alpha)$ to the first order,

$$2e\sin(c\theta - \alpha) = 2e\,\{\sin(cpt-\alpha) + 2e\sin(cpt-\alpha)\cos(cpt-\alpha)\}$$

to the second order,

$$= 2e\sin(cpt-\alpha) + 2e^2\sin 2(cpt-\alpha) \quad ,$$

and as θ and pt differ by a quantity of the first order, they may be used indiscriminately in terms of the second order, therefore

$$\left.\begin{aligned}\theta = pt + 2e\sin(cpt-\alpha) &+ \tfrac{5}{4}e^2\sin 2(cpt-\alpha)\\ &- \tfrac{1}{4}k^2\sin 2(gpt-\gamma)\\ &+ \tfrac{11}{8}m^2\sin\{(2-2m)pt - 2\beta\}\\ &+ \tfrac{15}{4}me\sin\{(2-2m-c)pt - 2\beta + \alpha\}\\ &- 3me'\sin(mpt + \beta - \zeta)\end{aligned}\right\} \quad \Theta'_2$$

52 In the value of u given in Art (48), substitute pt for θ in terms of the second order, and $pt + 2e\sin(cpt-\alpha)$ in the term of the first order, then

$$u = a\left\{\begin{aligned}1 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2 - e^2 &+ e\cos(cpt-\alpha) + e^2\cos 2(cpt-\alpha)\\ &- \tfrac{1}{4}k^2\cos 2(gpt-\gamma)\\ &+ m^2\cos\{(2-2m)pt - 2\beta\}\\ &+ \tfrac{15}{8}me\cos\{(2-2m-c)pt - 2\beta + \alpha\}\end{aligned}\right\} \quad U'_2,$$

the other terms in the value of u in Art (48), which were there retained only for the sake of subsequently finding t, being of the third order, are here omitted

53 Similarly, the expression for s becomes

$$s = k\sin\{(gpt-\gamma) + 2e\sin(cpt-\alpha)\} + \tfrac{3}{8}mk\sin\{(2-2m-g)pt - 2\beta + \gamma\},$$

or

$$s = k\left\{\begin{aligned}&\sin(gpt-\gamma)\\ &+ e\sin\{(g+c)pt - \alpha - \gamma\}\\ &- e\sin\{(g-c)pt + \alpha - \gamma\}\\ &+ \tfrac{3}{8}m\sin\{(2-2m-g)pt - 2\beta + \gamma\}\end{aligned}\right\} \quad . \; . \, S'_2$$

The expression for s is more complex in this form than when given in terms of the true longitude θ

54 If P be the moon's *mean* parallax, and Π the parallax at the time t,

$$\Pi = \frac{\text{radius of earth}}{\text{distance of } ☽} = \frac{R}{\dfrac{1}{u}\sqrt{(1+s^2)}} = Ru\left(1-\tfrac{1}{2}s^2\right) \text{ to the third order,}$$

$$= Ru\left\{1-\tfrac{1}{4}k^2+\tfrac{1}{4}k^2\cos 2(gpt-\gamma)\right\} \text{ to the second order,}$$

$$= Ra\left\{\begin{array}{l} 1-k^2-\tfrac{1}{2}m^2-e^2+e\cos(cpt-\alpha)+e^2\cos 2(cpt-\alpha) \\ \qquad\qquad\qquad + m^2\cos\{(2-2m)pt-2\beta\} \\ \qquad\qquad\qquad + \tfrac{15}{8}me\cos\{(2-2m-c)pt-2\beta+\alpha\}, \end{array}\right.$$

but $P =$ the portion which is independent of periodical terms,

$$= Ra\left(1-k^2-\tfrac{1}{2}m^2-e^2\right);$$

$$\text{therefore } \Pi = P\left\{\begin{array}{l} 1+e\cos(cpt-\alpha)+e^2\cos 2(cpt-\alpha) \\ + m^2\cos\{(2-2m)pt-2\beta\} \\ + \tfrac{15}{8}me\cos\{(2-2m-c)pt-2\beta+\alpha\} \end{array}\right.$$

neglecting terms of the third order

55 Here we terminate our approximations to the values of u, s, and θ If we wished to carry them to the third order, it would be necessary to include some terms of the fourth and fifth orders according to Art (29), and the values of P, T, and S, given in Art (23), would no longer be sufficiently accurate, but we should have to recur to more exact values, and from them obtain terms of an order beyond those already employed

If this be done, it is found that

$$\frac{P}{h^2u^2} = a\left\{1 + \qquad - \tfrac{9}{8}\,\frac{E-M}{E+M}\,m^2\,\frac{a'}{a}\cos(\theta-\theta') \qquad\right\},$$

$$\frac{T}{h^2u^3} = \qquad \cdot \; - \tfrac{3}{8}\,\frac{E-M}{E+M}\,m^2\,\frac{a'}{a}\sin(\theta-\theta') \qquad .$$

These terms of the fourth order become of the third order in the value of u, and therefore also of t, the coefficient of θ being near unity

We shall see further on (Appendix, Art 97), to what purpose a knowledge of the existence of these terms has been applied.

56 The process followed in the preceding pages is a sufficient clue to what must be done for a higher approximation

The coordinates u' and θ' of the sun's position are, by the theory of elliptic motion, known in terms of the time t, and t is given in terms of the longitude θ by the equation Θ_2 Hence u' and θ' can be obtained in terms of θ, but it will be necessary to take into account the slow progressive motion of the sun's perigee, which we have hitherto neglected This will be done by writing $c'\theta' - \zeta$ for $\theta' - \zeta$, c' being a quantity which differs very little from unity *

These values of u', θ', together with those of u and s in terms of θ, as given by U_2 and S_2, are then to be substituted in the corrected values of the forces, and thence in the differential equations The integrations being performed as before will give the values of u, s, and t in terms of θ to the third order, and from these, as in Arts (51), (52), and (53), may be obtained u, s, and θ in terms of t

57 More approximate values of c and g are obtained at the same time, by means of the coefficients of $\cos(c\theta - \alpha)$ and $\sin(g\theta - \gamma)$ in the differential equations, (see Appendix, Arts 94 and 95)

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* ‘En refléchissant sur les termes que doivent introduire toutes les quantités précédentes, on voit qu'il se peut glisser des cosinus de l'angle θ dont nous avons vu le dangereux effet d'amener dans la valeur de u des arcs au lieu de leurs cosinus, de tels termes viendront, par exemple, de la combinaison des cosinus de $(1-m)\theta$ avec des cosinus de $m\theta$

‘ Pour eviter cet inconvenient qui ôterait à la solution precédente l'avantage de convenir à un aussi grand nombre de revolutions qu'on voudrait, et la priverait de la simplicite et de l'universalite si précieuses en mathématiques, il faut commencer par en chercher la cause Or, on découvre facilement que ces termes ne viennent que de ce qu'on a supposé fixe l'apogee du soleil, ce qui n'est pas permis en toute rigueur, puisque quelque petite que soit sur cet astre l'action de la lune, elle n'en est pas moins reelle et doit lui produire un mouvement d'apogee quoique très lent a la verite ’

Clairaut, *Theorie de la Lune*, p 55 2me Edition

58 The values to the fourth order are then obtained from those of the third by continuing the same process, and so on to the fifth and higher orders, but the calculations are so complex that the approximations have not been carried beyond the fifth order, and already the value of θ in terms of t contains 128 periodical terms, without including those due to the disturbances produced by the planets The coefficients of these periodical terms are functions of m, e, e', $\frac{a'}{a}$, c, g, k, and are themselves very complicated under their literal forms. that of the term whose argument is twice the difference of the longitude of the sun and moon, for instance, is itself composed of 46 terms, combinations of the preceding constants.

See Pontécoulant, *Système du Monde*, tom IV p. 572.

CHAPTER V

NUMERICAL VALUES OF THE COEFFICIENTS

59 Having thus, from theory, obtained the *form* of the developments of the coordinates of the moon's position at any time, the next necessary step is the determination of the numerical values of the coefficients of the several terms

We here give three different methods which may be employed for that purpose, and these may, moreover, be combined according to circumstances

60 *First method* By particular observations of the sun and moon (*i e* by observations made when they occupy particular and selected positions), and also by observations separated by very long intervals, such, for instance, as ancient and modern eclipses, the values of the constants p, m, α, β, γ, ζ, which enter into the *arguments*, and of the additional ones which enter into the coefficients of the terms in the previous developments, may be obtained with great accuracy, and by their means, the coefficients themselves; c and g being also known in terms of the other constants

These may properly be called the *theoretical* values of the coefficients, the only recourse to observation being for the determination of the numerical values of the elements

61 *Second method.* Let the constants which enter into the *arguments* be determined as in the first method; and let a large number of observations be made, from each of which a value of the true longitude, latitude, or parallax is obtained, together

with the corresponding value of t reckoned from the fixed epoch when the mean longitude is zero Let these corresponding values be substituted in the equations, each observation thus giving rise to a relation between the unknown constant coefficients.

A very great number of equations being thus obtained, they are then, by the method of least squares or some analogous process, reduced to as many as there are coefficients to be determined The solution of these simple equations will give the required values

This method, however, would scarcely be practicable in a high order of approximation For instance, in the fifth order, as stated in Art (58), each of the numerous equations would consist of 130 terms, and these would have to be reduced to 129 equations of 130 terms each

62. *Third method* When the constants which enter into the arguments have been determined by the first method, we may obtain any one of the coefficients independently of all the others by the following process, provided the number of observations be very great

Let the form of the function be

$$V = A + B\sin\theta + C\sin\phi + \&c,$$

and let it be required to determine the constants A, B, C, &c. separately; θ, ϕ, &c being functions of the time.

Let the results of a great number of observations corresponding to values θ_1, θ_2, θ_3, &c, ϕ_1, ϕ_2, ϕ_3, &c, be V_1, V_2, V_3, &c ; so that

$$V_1 = A + B\sin\theta_1 + C\sin\phi_1 + \&c,$$

$$V_2 = A + B\sin\theta_2 + C\sin\phi_2 + \&c.,$$

$$V_3 = A + B\sin\theta_3 + C\sin\phi_3 + \&c,$$

$$V_n = A + B\sin\theta_n + C\sin\phi_n + \&c.$$

Now, n being *very great*, we may assume that the periodical terms will, on the whole, be as often positive as negative; and therefore, that if we add all the equations together these terms will cancel one another;

therefore $$A = \frac{V_1 + V_2 + V_3 + \qquad + V_n}{n},$$

which determines the non-periodic part of the function

To determine B Let the observations be divided into two sets separating the positive and negative values of $\sin\theta$; then the other periodical terms, not having the same period, may be considered as cancelling themselves in adding up the terms of each set Let there be r terms in the first set and s terms in the second, and let V', V'', $\qquad V^r$ be the values of V corresponding to positive values of $\sin\theta$, which values we may assume to be uniformly distributed from $\sin 0$ to $\sin\pi$, and therefore to be $\sin\delta\theta$, $\sin 2\delta\theta$, $\qquad \sin r\,\delta\theta$, where $r\,\delta\theta = \pi$

And, again, let $V_{,}$, $V_{,,}$, $V_{,,,}$, $\quad . V_{s}$, be the values of V corresponding to the negative values of $\sin\theta$, viz. $\sin(-\Delta\theta)$, $\sin(-2\Delta\theta)$, $\sin(-s\,\Delta\theta)$, where $s\,\Delta\theta = \pi$ Then,

$$V' = A + B\sin\delta\theta + C\sin\phi' + \quad ,$$
$$V'' = A + B\sin 2.\delta\theta + C\sin\phi'' + \,..,$$
$$V^r = A + B\sin r\,\delta\theta + C\sin\phi^r + \quad ;$$

therefore

$$V' + V'' + .\;\; + V^r = r.A + B\,\Sigma_0^{\pi}(\sin\theta)$$
$$= rA + \frac{B}{\delta\theta}\,\Sigma_0^{\pi}(\sin\theta.\delta\theta);$$

therefore

$$\frac{V' + V'' + \quad + V^r}{r} = A + \frac{B}{\pi}\int_0^{\pi}\sin\theta.d\theta$$
$$= A + \frac{2B}{\pi}.$$

$$V_{,} = A - B\sin\Delta\theta + C\sin\phi_{,} + \,.\,,$$
$$V_{,,} = A - B\sin 2\,\Delta\theta + C\sin\phi_{,,} + \quad ,$$
$$V_{s} = A - B\sin s.\Delta\theta + C\sin\phi_{s} + \quad ;$$

therefore

$$V_{,} + V_{,,} + .\;\; + V_{s} = s\,A - B\,\Sigma_0^{\pi}(\sin\theta)$$
$$= sA - \frac{B}{\Delta\theta}\,\Sigma_0^{\pi}(\sin\theta\,\Delta\theta),$$

therefore

$$\frac{V_{,} + V_{,,} + \quad + V_{s}}{s} = A - \frac{B}{\pi}\int_0^{\pi}\sin\theta\,d\theta$$
$$= A - \frac{2B}{\pi};$$

therefore $$B = \frac{\pi}{4}\left(\frac{V' + V'' + ..\; V^r}{r} - \frac{V_{,} + V_{,,} + .\;. + V_{s}}{s}\right);$$

and in a similar manner may each of the coefficients be independently determined *

For further remarks on this method, see Appendix, Art (104)

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* If r and s are not sufficiently great to allow us to substitute $\int_0^\pi \sin\theta\, d\theta$ for $\Sigma_0^\pi \sin\theta\,\delta\theta$, we must proceed as follows

$$V' + V'' + \quad + V^r = rA + B(\sin\delta\theta + \sin 2\delta\theta + \quad + \sin r\delta\theta)$$

$$= rA + B\frac{\sin\frac{1}{2}(r+1)\,\delta\theta\,\sin\frac{1}{2}r\,\delta\theta}{\sin\frac{1}{2}\delta\theta}$$

$$= rA + B\frac{\cos\frac{1}{2}\delta\theta}{\sin\frac{1}{2}\delta\theta},$$

$$\frac{V' + V'' + \quad + V^r}{r} = A + \frac{B}{r}\,\frac{1-\frac{1}{8}\delta\theta^2}{\frac{1}{2}\delta\theta - \frac{1}{48}\delta\theta^3}\text{, nearly,}$$

$$= A + \frac{2B}{\pi}\,\frac{1-\frac{1}{8}\frac{\pi^2}{r^2}}{1-\frac{1}{24}\frac{\pi^2}{r^2}}$$

$$= A + \frac{2B}{\pi}\left(1-\frac{1}{12}\frac{\pi^2}{r^2}\right)$$

Similarly, $$\frac{V_{,} + V_{,,} + \quad + V_s}{s} = A - \frac{2B}{\pi}\left(1-\frac{1}{12}\frac{\pi^2}{s^2}\right),$$

therefore $$B = \frac{\pi}{4\left\{1-\frac{\pi^2}{24}\left(\frac{1}{r^2}+\frac{1}{s^2}\right)\right\}}\left(\frac{V'+V''+\quad+V^r}{r} - \frac{V_{,}+V_{,,}+\quad+V_s}{s}\right)$$

$$= \frac{\pi}{4}\left\{1+\frac{\pi^2}{24}\left(\frac{1}{r^2}+\frac{1}{s^2}\right)\right\}\left(\frac{V'+V''+\quad+V^r}{r} - \frac{V_{,}+V_{,,}+\quad+V_s}{s}\right)$$

CHAPTER VI.

PHYSICAL INTERPRETATION

63 The solution of the problem which is the object of the Lunar Theory may now be considered as effected; that is, we have obtained equations which enable us to assign the moon's position in the heavens at any given time to the second order of approximation; we have explained how the numerical values of the coefficients in these equations may be determined from observation; and we have, moreover, shewn how to proceed in order to obtain a higher approximation [*]

It will, however, be interesting to discuss the results we have arrived at, to see whether they will enable us to form some idea of the nature of the moon's complex motion, and also whether they will explain those inequalities or departures from uniform circular motion which ancient astronomers had observed, but which, until the time of Newton, were so many unconnected phenomena, or, at least, had only such arbitrary connexions as the astronomers chose to assign, by grafting one eccentric or epicycle on another as each newly discovered inequality seemed to render it necessary.

It is true that our expressions, composed of periodic terms, are nothing more than translations into analytical language of the epicycles of the ancients;[†] but they are evolved directly from the fundamental laws of force and motion, and as many new terms as we please may be obtained by carrying on the same process; whereas the epicycles of Hipparchus and his followers were the result of numerous and laborious observations and comparisons of observations, each epicycle being introduced to correct its predecessor when this one was found inadequate to give the position of the body at all times: just as with us, the terms of the second order correct the rough results given by those of the first, the terms of the third order correct those of the second, and so on But it is impossible to conceive that observation alone could have detected *all* those minute irregularities which theory makes known to us in the terms of the third and higher orders, even supposing our instruments far more perfect than they are, and it will always be a subject of admiration and surprise, that Tycho, Kepler, and their predecessors should have been able to *feel* their way so far among the Lunar inequalities, with the means of observation they possessed

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[*] The means of taking into account the ellipsoidal figure of the earth and the disturbances produced by the planets, are too complex to form part of an introductory treatise For information on these points reference may be made to Airy's *Figure of the Earth* Pontécoulant's *Système du Monde*," vol IV

[†] See Whewell's "History of the Inductive Sciences"

LONGITUDE OF THE MOON

64 We shall firstly discuss the expression for the moon's longitude, as found Art (51)

$$\begin{aligned}\theta = pt &+ 2e \sin(cpt - \alpha) + \tfrac{5}{4}e^2 \sin 2(cpt - \alpha)\\ &+ \tfrac{15}{4}me \sin\{(2 - 2m - c)\,pt - 2\beta + \alpha\}\\ &+ \tfrac{11}{8}m^2 \sin\{(2 - 2m)\,pt - 2\beta\}\\ &- 3me' \sin(mpt + \beta - \zeta)\\ &- \tfrac{1}{4}k^2 \sin 2(gpt - \gamma)\end{aligned}$$

The mean value of θ is pt; and in order to judge of the effect of any of the small terms, we may consider them one at a time as a correction on this mean value pt, or we may select a combination of two or more to form this correction.

We shall have instances of combinations in explaining the *elliptic inequality* and the *evection*, Arts (66) and (70); but in the remaining inequalities each term of the expression will form a correction to be considered by itself

65 Neglecting all the periodical terms, we have

$$\theta = pt,$$

$$\frac{d\theta}{dt} = p,$$

which indicates uniform angular velocity; and as, to the same order, the value of u is constant, the two together indicate that the moon moves uniformly in a circle, the period of a revolution being $\frac{2\pi}{p}$, which is, therefore, the expression for a mean sidereal month, or about $27\frac{1}{3}$ days.*

The value of p is, according to Art (50), given by

$$\frac{1}{p} = \frac{1}{ha^2}\left(1 + \tfrac{3}{2}k^2 + m^2 + \tfrac{3}{2}e^2\right),$$

and as m is due to the disturbing action of the sun, we see that the mean angular velocity is less, and therefore the mean periodic time greater than if there were no disturbance

Elliptic inequality or Equation of the Centre.

66. We shall next consider the effect of the first three terms together· the effect of the second alone, as a correction of pt, will be discussed in the Historical Chapter, Art. (109)

$$\theta = pt + 2e\sin(cpt - \alpha) + \tfrac{5}{4}e^2\sin 2(cpt - \alpha),$$

which may be written

$$\theta = pt + 2e\sin[pt - \{\alpha + (1-c)pt\}] + \tfrac{5}{4}e^2\sin 2[pt - \{\alpha + (1-c)pt\}].$$

But the connexion between the longitude and the time in an ellipse described about a centre of force in the focus, is, Art. (13), to the second order of small quantities.

$$\theta = nt + 2e\sin(nt - \alpha') + \tfrac{5}{4}e^2\sin 2(nt - \alpha'),$$

---

* The accurate value was 27d 7h 43m 11 261s in the year 1801 See Art (99)

where n is the mean motion, e the eccentricity, and α' the longitude of the apse.*

Hence, the terms we are now considering indicate motion in an ellipse; the mean motion being p, the eccentricity e, and the longitude of the apse $\alpha + (1-c)\,pt$, that is, the apse has a progressive motion in longitude, uniform, and equal to $(1-c)p$

67. The two terms $2e\sin(cpt-\alpha) + \frac{5}{4}e^2\sin 2(cpt-\alpha)$ constitute the *elliptic inequality*, and their effect may be further illustrated by means of a diagram

Let the full line AMB (fig 8) represent the moon's orbit about the earth E, when the time t commences, that is, when the moon's *mean* place is in the prime radius $E\Upsilon$ from which the longitudes are reckoned

The angle ΥEA, the longitude of the apse, is then α At the time t, when the moon's mean longitude is $\Upsilon EM = pt$, the apse line will have moved in the same direction through the angle $AEA' = (1-c)\,\Upsilon EM$, and the orbit will have taken the position indicated by the dotted ellipse, and the true place of the moon in this orbit, so far as these two terms are concerned, will be m, where

$$\begin{aligned} MEm &= 2e\sin(cpt-\alpha) + \tfrac{5}{4}e^2\sin 2\,(cpt-\alpha) \\ &= 2e\sin A'EM \quad + \tfrac{5}{4}e^2\sin 2A'EM \\ &= 2e\sin A'EM(1 + \tfrac{5}{4}e\cos A'EM)\,, \end{aligned}$$

which, since e is about $\frac{1}{20}$, is positive from perigee to apogee, and therefore the *true* place before the *mean*, and the contrary from apogee to perigee: at the apses the places will coincide

68. The angular velocity of the apse is $(1-c)p$, or, if for c we put the value found in Art (48), the velocity will be $\frac{3}{4}m^2p$. Hence, while the moon describes 360°, the apse should describe $\frac{3}{4}m^2\,360° = 1\frac{3}{5}°$ nearly, m being about $\frac{1}{13}$

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[*] The epoch ε which appears in the expression of Art (13) is here omitted, a proper assumption for the origin of t, as explained in Art (34), enabling us to avoid the ε

But Hipparchus had found, and all modern observations confirm his result, that the motion of the apse is about 3° in each revolution of the moon See Art (112)

This difference arises from our value of c not being represented with sufficient accuracy by $1 - \frac{3}{4}m^2$

Newton himself was aware of this apparent discrepancy between his theory and observation, and we are led, by his own expressions (Scholium to Prop 35, lib III in the first edition of the *Principia*), to conclude that he had got over the difficulty This is rendered highly probable when we consider that he had solved a somewhat similar problem in the case of the node; but he has nowhere given a statement of his method and Clairaut, to whom we are indebted for the solution, was on the point of publishing a new hypothesis of the laws of attraction, in order to account for it, when it occurred to him to carry the approximations to the third order, and he found the next term in the value of c nearly as considerable as the one already obtained See Appendix, Art (94).

$$c = 1 - \tfrac{3}{4}m^2 - \tfrac{225}{32}m^3,$$

$$1 - c = \tfrac{3}{4}m^2 + \tfrac{225}{32}m^3 = \tfrac{3}{4}m^2\left(1 + \tfrac{75}{8}m\right),$$

$$\therefore \quad (1-c)\,360^\circ = \left(1 + \tfrac{75}{104}\right) \text{ (value found previously)}$$

$$= 2\tfrac{3}{4}^\circ \text{ nearly},$$

thus reconciling theory and observation, and removing what had proved a great stumbling-block in the way of all astronomers *

When the value of c is carried to higher orders of approximation, the most perfect agreement is obtained

The motion of the apse line is considered by Newton in his *Principia*, lib I, Prop 66, Cor. 7.

Evection

69. The next term in the value of θ is

$$+ \tfrac{15}{4}me \sin\{(2 - 2m - c)pt - 2\beta + \alpha\}$$

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* See Dr Whewell's *Bridgewater Treatise*

We shall consider this term in two different ways.

Firstly, by itself, as forming a correction on pt.

$$\theta = pt + \tfrac{15}{4}me \sin\{(2-2m-c)pt - 2\beta + \alpha\}$$

Let $\text{☽} = pt$ = moon's mean longitude at time t,

$\bigcirc = mpt + \beta$ = sun's .. ,

$\alpha' = (1-c)pt + \alpha$ = mean longitude of apse ,

then

$$\theta = pt + \tfrac{15}{4}me \sin[2\{pt - (mpt+\beta)\} - \{pt - \overline{(1-c)pt+\alpha}\}]$$
$$= pt + \tfrac{15}{4}me \sin[2(\text{☽} - \bigcirc) - (\text{☽} - \alpha')]$$

The effect of this term will therefore be as follows.

In syzygies $\theta = pt - \tfrac{15}{4}me \sin(\text{☽} - \alpha')$;

or the true place of the moon will be before or behind the mean, according as the moon, at the same time, is between apogee and perigee or between perigee and apogee.

In quadratures $\theta = pt + \tfrac{15}{4}me \sin(\text{☽} - \alpha')$,

and the circumstances will be exactly reversed

In both cases, the correction will vanish when the apse happens to be in syzygy or quadrature at the same time as the moon

In intermediate positions, the nature of the correction is more complex, but it will always vanish when the sun's longitude is equal to the arithmetical mean between those of the moon and apse, or when it differs from it by any multiple of 90°; for if $\bigcirc = \frac{\text{☽} + \alpha'}{2} - r\,90°$,

$$\begin{aligned}\sin[2(\text{☽} - \bigcirc) - (\text{☽} - \alpha')] &= \sin(\text{☽} + \alpha' - 2\bigcirc)\\ &= \sin r\,180°\\ &= 0.\end{aligned}$$

70. The other and more usual method of considering the effect of this term is in combination with the two terms of the elliptic inequality, as follows:

To determine the change in the position of the apse and in the eccentricity of the moon's orbit produced by the evection.

Taking the elliptic inequality and the evection together, we have

$$\theta = pt + 2e\sin(cpt-\alpha) + \tfrac{5}{4}e^2\sin 2(cpt-\alpha)$$
$$+ \tfrac{15}{4}me\sin\{(2-2m-c)pt-2\beta+\alpha\}$$

Let α' be the longitude of the apse at time t on supposition of uniform progression,

$\bigcirc$ . . . . . . sun ;

whence
$$\alpha' = (1-c)pt + \alpha,$$
$$\bigcirc = mpt + \beta$$

And the above may be written

$$\theta = pt + 2e\sin(cpt-\alpha) + \tfrac{5}{4}e^2\sin 2(cpt-\alpha)$$
$$+ \tfrac{15}{4}me\sin\{cpt-\alpha+2(\alpha'-\bigcirc)\},$$

and the second and fourth terms may be combined into one,

$$2E\sin(cpt-\alpha+\delta),$$

if
$$E\cos\delta = e + \tfrac{15}{8}me\cos 2(\alpha'-\bigcirc),$$
$$E\sin\delta = \tfrac{15}{8}me\sin 2(\alpha'-\bigcirc),$$

whence
$$\tan\delta = \frac{\tfrac{15}{8}m\sin 2(\alpha'-\bigcirc)}{1+\tfrac{15}{8}m\cos 2(\alpha'-\bigcirc)},$$
$$E^2 = e^2\{1+\tfrac{15}{8}m\cos 2(\alpha'-\bigcirc)\}^2 + e^2\{\tfrac{15}{8}m\sin 2(\alpha'-\bigcirc)\}^2;$$

or, approximately,
$$\delta = \tfrac{15}{8}m\sin 2(\alpha'-\bigcirc),$$
$$E = e\{1+\tfrac{15}{8}m\cos 2(\alpha'-\bigcirc)\}.$$

The term $\tfrac{5}{4}e^2\sin 2(cpt-\alpha)$ will, therefore, to the second order, be expressed by

$$\tfrac{5}{4}E^2\sin 2(cpt-\alpha+\delta),$$

and the longitude becomes

$$\theta = pt + 2E\sin(cpt-\alpha+\delta) + \tfrac{5}{4}E^2\sin 2(cpt-\alpha+\delta),$$

or
$$\theta = pt + 2E\sin(pt-\alpha'+\delta) + \tfrac{5}{4}E^2\sin 2(pt-\alpha'+\delta);$$

but the last two terms constitute elliptic inequality in an orbit whose eccentricity is E and longitude of the apse $\alpha'-\delta$; therefore the evection, taken in conjunction with elliptic inequality, has the effect of rendering the eccentricity of the moon's orbit variable, increasing it by $\frac{15}{8}me$ when the apse-line is in syzygy, and diminishing it by the same quantity when the apse-line is in quadrature; the general expression for the increment being

$$\tfrac{15}{8}me \cos 2(\alpha' - \odot)$$

And another effect of this term is, to diminish the longitude of the apse, calculated on the supposition of its uniform progression, by the quantity $\delta = \frac{15}{8}m \sin 2(\alpha' - \odot)$; so that the apse is behind its mean place from syzygy to quadrature, and before it from quadrature to syzygy *

The cycle of these changes will evidently be completed in the period of half a revolution of the sun with respect to the apse, or in about $\frac{9}{16}$ of a year

71. The period of the evection itself, considered independently of its effect on the orbit, is the time in which the argument $(2 - 2m - c)\,pt - 2\beta + \alpha$ will increase by 2π Therefore period of evection

$$= \frac{2\pi}{(2-2m-c)\,p} = \frac{\text{mean sidereal month}}{2 - 2m - c}$$

$$= \frac{\text{mean sidereal month}}{1 - 2m + \frac{3}{4}m^2} = \frac{27\frac{1}{3}\text{ days}}{1 - \frac{2}{13}}, \text{ nearly},$$

$$= 31\tfrac{1}{2}\text{ days, nearly.}\dagger$$

Newton has considered the evection, so far as it arises from the central disturbing force, in Prop. 66, Cor. 9 of the *Principia*

Variation

72 *To explain the physical meaning of the term*

$$\tfrac{11}{8}m^2 \sin\{(2-2m)\,pt - 2\beta\},$$

in the expression for the moon's longitude,

$$\theta = pt + \tfrac{11}{8}m^2 \sin\{(2-2m)\,pt - 2\beta\}.$$

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* The change of eccentricity and the variation in the motion of the apse follow the same law as the abscissa and ordinate of an ellipse referred to its centre for if $E - e = x$ and $\delta = y$, then

$$\frac{x^2}{(\frac{15}{8}me)^2} + \frac{y^2}{(\frac{15}{8}m)^2} = 1$$

† The accurate value is 31 8119 days

Let ☽ represent the moon's mean longitude at time t,

○ sun's

therefore

$$☽ = pt,$$
$$○ = mpt + \beta,$$

and the value of θ becomes

$$\theta = pt + \tfrac{11}{8} m^2 \sin 2(☽ - ○),$$

which shews that from syzygy to quadrature, the moon's true place is before the mean, and behind it from quadrature to syzygy; the maximum difference being $\frac{11}{8}m^2$ in the octants.

The angular velocity of the moon, so far as this term is concerned, is

$$\frac{d\theta}{dt} = p + \tfrac{11}{4}(1-m)m^2 p \cos 2(☽ - ○),$$
$$= p\{1 + \tfrac{11}{4} m^2 \cos 2(☽ - ○)\}, \text{ nearly},$$

which exceeds the mean angular velocity p at syzygies, is equal to it in the octants, and less in the quadratures.

This inequality has been called the *Variation*, its period is the time in which the argument $(2-2m)pt - 2\beta$ will increase by 2π,

$$\text{period of variation} = \frac{2\pi}{(2-2m)p} = \frac{\text{mean synodical month}}{2}$$
$$= 14\tfrac{3}{4} \text{ days, nearly.*}$$

73 The quantity $\frac{11}{8}m^2$ is only the first term of an endless series which constitutes the coefficient of the variation, the other terms being obtained by carrying the approximation to a higher order. It is then found that the next term in the coefficient is $\frac{59}{12}m^3$, which is about $\frac{3}{11}$ of the first term; and as there are several other important terms, it is only by carrying the approximation to a high order (the 5th at least) that the value of this coefficient can be obtained with sufficient accuracy from theory. In fact, $\frac{11}{8}m^2$ would give a coefficient of 28′ 32″ only, whereas the accurate value is found to be 39′ 30″.

The same remark applies also to the coefficients of all the other terms.

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* The accurate value is 14·765294 days.

74 As far as terms of the second order, the coefficient of the variation is independent of e the eccentricity, and k the inclination of the orbit It would therefore be the same in an orbit originally circular, whose plane coincided with the plane of the ecliptic it is thus that Newton has considered it. *Princip.* Prop 66, Cor 3, 4, and 5

Annual Equation

75 *To explain the physical meaning of the term*

$$-3me' \sin(mpt+\beta-\zeta)$$

in the expression for the moon's longitude

$$\begin{aligned} \theta &= pt - 3me' \sin(mpt + \beta - \zeta), \\ &= pt - 3me' \sin(\text{longitude of sun} - \text{longitude of sun's perigee}), \\ &= pt - 3me' \sin(\text{sun's anomaly}) \end{aligned}$$

Hence, while the sun moves from perigee to apogee, the true place of the moon will be behind the mean, and from apogee to perigee, before it The period being an anomalistic year, the effect is called *Annual Equation.*

Differentiating θ we get

$$\frac{d\theta}{dt} = p\{1 - 3m^2e' \cos(\text{sun's anomaly})\}$$

Hence, so far as this inequality is concerned, the moon's angular velocity is least when the sun is in perigee, that is *at present* about the 1st of January, and greatest when the sun is in apogee, or about the 1st of July

The annual equation is, to this order, independent of the eccentricity and inclination of the moon's orbit, and therefore, like the variation, would be the same in an orbit originally circular *Vide* Newton, *Principia*, Prop. 66, Cor. 6

Reduction

76. Before considering the effect of the term $-\frac{k^2}{4}\sin 2(gpt-\gamma)$, which, as we shall see Art. (82), is very nearly equal to the difference between the longitude in the orbit and the longitude

in the ecliptic, it will be convenient to examine the expression for the latitude of the moon, and to see how the motion of the node is connected with the value of g.

LATITUDE OF THE MOON

77. The expression Art (49), found for the tangent of the latitude,* is

$$s = k \sin(g\theta - \gamma) + \tfrac{3}{8}mk \sin\{(2 - 2m - g)\,\theta - 2\beta + \gamma\}$$

If we reject all small terms, we have

$$s = 0,$$

or the orbit of the moon coinciding with the ecliptic, which is a first rough approximation to its true position

78 Taking the first term of the expansion

$$s = k \sin(g\theta - \gamma),$$

we may write it

$$s = k \sin[\theta - \{\gamma - (g - 1)\,\theta\}]$$

Let ΥNm be the ecliptic (fig 9), N the moon's node when her true longitude is zero, and let M be the position of the moon at time t, m her place referred to the ecliptic;

therefore $\Upsilon N = \gamma$, $\Upsilon m = \theta$, $\tan Mm = s$

Take $NN' = (g - 1)\,\theta$ in a retrograde direction, and join MN' by an arc of great circle,

then $\sin N'm = \tan Mm \cot MN'm,$

or $\sin[\theta - \{\gamma - (g - 1)\,\theta\}] = s \cot MN'm,$

which, compared with the value of s given above, shews that $MN'm = \tan^{-1}k$ is constant, and therefore the term $k \sin(g\theta - \gamma)$ indicates that the moon moves in an orbit inclined at an angle $\tan^{-1}k$ to the ecliptic, and whose node regredes along the ecliptic with the velocity $(g - 1)\dfrac{d\theta}{dt}$, or with a mean velocity $(g - 1)\,p$

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* This expression for the tangent of the latitude is more convenient than that which gives it in terms of the mean longitude, Art (53) on account of the less number of terms involved. See Pontécoulant, vol IV, p. 630.

79. Hence the period of a revolution of the nodes $= \dfrac{2\pi}{(g-1)p}$

$$= \frac{\text{one sidereal month}}{g-1},$$

but, from Art (49), the value of $g = 1 + \frac{3}{4}m^2$;

therefore period of revolution of nodes $= \dfrac{\text{one sidereal month}}{\frac{3}{4}m^2}$

$$= 6511 \text{ days, nearly}$$

This will, for the same reason as in the case of the apse, Art (68), be modified when we carry the approximation to a higher degree, this value of g is, however, much more accurate than the corresponding value of c, for the third term of g is small; the value to the third order being (see Appendix, Art. 95)

$$g = 1 + \tfrac{3}{4}m^2 - \tfrac{9}{32}m^3,$$

and the period of revolution of the nodes $= \dfrac{\text{one sidereal month}}{\frac{3}{4}m^2(1-\frac{3}{8}m)}$

$$= 6705 \text{ days, nearly.}$$

This is not far from the accurate value as given by observation, and when the approximation to the value of g is carried to a higher order, the agreement is nearly perfect

The true value is 6793 39 days, that is about 18 yrs, 7 mo

Evection in Latitude

80. *To explain the variation of the inclination and the irregularity in the motion of the node expressed by the term*

$$+ \tfrac{3}{8}mk \sin\{(2-2m-g)\,\theta - 2\beta + \gamma\}.$$

This term, as a correction on the preceding, is analogous to the evection as a correction on the elliptic inequality.

Taking the two terms together,

$$s = k\sin(g\theta - \gamma) + \tfrac{3}{8}mk \sin\{(2-2m-g)\,\theta - 2\beta + \gamma\}$$

Let $☽$ = longitude of moon $= \theta$,

$\bigcirc$ = ... sun $= m\theta + \beta$,

Ω = . . node $= \gamma - (g-1)\,\theta$,

therefore $s = k\sin(☽ - \Omega) + \tfrac{3}{8}mk\sin\{☽ - \Omega - 2(\bigcirc - \Omega)\}$

Now these two terms may be combined into one

$$s = K \sin(\text{☽} - \Omega - \delta),$$

if

$$K \cos\delta = k + \tfrac{3}{8} mk \cos 2(\bigcirc - \Omega),$$

$$K \sin\delta = \tfrac{3}{8} mk \sin 2(\bigcirc - \Omega),$$

whence

$$\tan\delta = \frac{\tfrac{3}{8} m \sin 2(\bigcirc - \Omega)}{1 + \tfrac{3}{8} m \cos 2(\bigcirc - \Omega)},$$

$$K^2 = k^2\{1 + \tfrac{3}{8} m \cos 2(\bigcirc - \Omega)\}^2 + k^2\{\tfrac{3}{8} m \sin 2(\bigcirc - \Omega)\}^2,$$

or approximately,

$$\delta = \tfrac{3}{8} m \sin 2(\bigcirc - \Omega),$$

$$K = k\{1 + \tfrac{3}{8} m \cos 2(\bigcirc - \Omega)\},$$

but the equation

$$s = K \sin(\text{☽} - \Omega - \delta)$$

represents motion in an orbit inclined at an angle $\tan^{-1} K$ to the ecliptic, and the longitude of whose node is $\Omega + \delta$.

This term has therefore the following effects

1st. The inclination of the moon's orbit is variable, its tangent increases by $\tfrac{3}{8} mk$ when the nodes are in syzygies, and decreases by the same quantity when they are in quadrature; the general expression for the increase being $\tfrac{3}{8} mk \cos 2(\bigcirc - \Omega)$.

2nd The longitude of the node, calculated on supposition of a uniform regression, is increased by $\delta = \tfrac{3}{8} m \sin 2(\bigcirc - \Omega)$, so that the node is before its mean place while moving from syzygy to quadrature and behind it from quadrature to syzygy.

Principia, book III, props 33 and 35.

The cycle of these changes will be completed in the period of half a revolution of the sun with respect to the node, that is, in 173 21 days, not quite half-a-year

81 The tangent of the latitude has here been obtained; if we wish to have the latitude itself it will be given by the formula

$$\text{latitude} = s - \tfrac{1}{3} s^3 + \tfrac{1}{5} s^5 - \&\text{c},$$

which, to the degree of approximation adopted, will clearly be the same as s.

Reduction

82 We may now consider the term which we had neglected Art 76) in the expression for the longitude, namely,

$$-\tfrac{1}{4}k^2 \sin 2(gpt-\gamma).$$

Let N (fig 10) be the position of the node when the moon's ongitude is θ, M the place of the moon, m the place referred o the ecliptic.

Therefore
$$\Upsilon m = \theta,$$
$$\Upsilon N = \gamma - (g-1)\,\theta,$$
$$Nm = g\theta - \gamma,$$
$$\tan N = k$$

The right-angled spherical triangle NMm gives

$$\cos N = \frac{\tan Nm}{\tan NM};$$

herefore
$$\frac{1-\cos N}{1+\cos N} = \frac{\tan NM - \tan Nm}{\tan NM + \tan Nm},$$

r
$$\tan^2\frac{N}{2} = \frac{\sin(NM-Nm)}{\sin(NM+Nm)},$$

r, since both N and $NM - Nm$ are small,

$$\frac{\tan^2 N}{4} = \frac{NM-Nm}{\sin 2Nm} \text{ approximately};$$

herefore $NM - Nm = \frac{1}{4}k^2 \sin 2(g\theta-\gamma) = \frac{1}{4}k^2 \sin 2(gpt-\gamma)$, nearly

Hence this term, which is called the *reduction*, is approximately the difference between the longitude in the orbit and he longitude in the ecliptic

RADIUS VECTOR

83 *To explain the physical meaning of the terms in the alue of* u.

We shall, for the explanation, make use of the formula hich gives the value of u in terms of the true longitude, rt (48).

Firstly, neglecting the periodical terms, we have for the mean value

$$u = a(1 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2)$$

The term $-\tfrac{1}{2}m^2$, which is a consequence of the disturbing effect of the sun, shews that the mean value of the moon's radius vector, and therefore the orbit itself, is larger than if there were no disturbance

Elliptic Inequality

84 To explain the effect of the term of the *first order*,

$$\begin{aligned} u &= a\{1 + e\cos(c\theta - \alpha)\}, \\ &= a[1 + e\cos\overline{\theta - \{\alpha + (1 - c)\,\theta\}}] \end{aligned}$$

This is the *elliptic inequality*, and indicates motion in an ellipse whose eccentricity is e and longitude of the apse $\alpha + (1 - c)\,\theta$, and the same conclusion is drawn with respect to the motion of the apse as in Art (66)

Evection

85 *To explain the physical meaning of the term*

$$\tfrac{15}{8}mea\cos\{(2 - 2m - c)\,\theta - 2\beta + \alpha\}$$

This, as in the case of the corresponding term in the longitude, is best considered in connexion with the elliptic inequality, and exactly the same results will follow

Thus calling $\rightmoon$, $\bigcirc$, and α' the true longitudes of the moon, sun, and apse, the latter calculated on supposition of uniform motion, these two terms may be written,

$$\begin{aligned} u &= a[1 + e\cos(\rightmoon - \alpha') + \tfrac{15}{8}me\cos\{\rightmoon - \alpha' + 2(\alpha' - \bigcirc)\}] \\ &= a[1 + E\cos(\rightmoon - \alpha' + \delta)]; \end{aligned}$$

where

$$\begin{aligned} E\cos\delta &= e + \tfrac{15}{8}me\cos 2(\alpha' - \bigcirc), \\ E\sin\delta &= \tfrac{15}{8}me\sin 2(\alpha' - \bigcirc). \end{aligned}$$

These are identical with the equations of Art (70)

Variation

86. *To explain the effect of the term* $m^2a\cos\{(2-2m)\,\theta-2\beta\}$,

$$u = a[1+m^2\cos\{(2-2m)\,\theta-2\beta\}]$$
$$= a[1+m^2\cos 2(☽-☉)].$$

As far as this term is concerned, the moon's orbit would be an oval having its longest diameter in quadratures and least in syzygies *Principia*, lib I, prop 66 cor 4

The ratio of the axes of the oval orbit will be

$$\frac{1+m^2}{1-m^2}=\tfrac{90}{89} \text{ nearly, } m \text{ being } \cdot 0748.$$

See *Principia*, lib. III, prop 28

Reduction

87 The last important periodical term in the value of u is

$$-\frac{ak^2}{4}\cos 2(g\theta-\gamma).$$

This, when increased by a *constant*, is approximately the difference between the values of u in the orbit and in the ecliptic.

For if u_1 be the reciprocal of the value of the radius vector in the orbit,

$$u_1 = u\cos(\text{latitude}),$$
$$=\frac{u}{\sqrt{(1+s^2)}}=u(1-\tfrac{1}{2}s^2),\text{ nearly,}$$

therefore $u-u_1=\frac{1}{2}us^2=\frac{1}{2}ak^2\sin^2(g\theta-\gamma)$

$$=\tfrac{1}{4}ak^2-\tfrac{1}{4}ak^2\cos 2(g\theta-\gamma)$$
$$=\text{const } -\tfrac{1}{4}ak^2\cos 2(g\theta-\gamma)$$

88 The remaining terms in the value of u are of the third order, and therefore very small: one of these corresponds to the *annual equation* in longitude Art (75), where it is of the second order, having increased in the course of integration

Periodic time of the Moon

89 We have seen, Art (65), that the periodic time of the moon is greater than if there were no disturbing force, but this refers to the *mean* periodic time estimated on an interval of a great number of years, so that the circular functions in the expression are then extremely small compared with the quantity pt which has uniformly increased

When, however, we consider only a few revolutions, these terms may not all be neglected. The *elliptic inequality* and the *evection* go through their values in about a month, the *variation* and *reduction* in about half-a-month, their effects, therefore, on the length of the period can scarcely be considered, as they will increase one portion and then decrease another of the *same* month

But the *annual equation* takes one year to go through its cycle, and, during this time, the moon has described thirteen revolutions, hence, fluctuations may, and, as we shall now shew, do take place in the lengths of the sidereal months during the year.

We have, considering only the annual equation, Art (75),

$$pt = \theta + 3me' \sin(m\theta + \beta - \zeta).$$

Let T be the length of the period, then when θ is increased by 2π, t becomes $t + T$;

therefore $p(t + T) = 2\pi + \theta + 3me' \sin(2m\pi + m\theta + \beta - \zeta)$,

whence $\qquad pT = 2\pi + 6me' \sin m\pi \cos(m\pi + m\theta + \beta - \zeta)$;

therefore $\qquad T = \text{mean period} + \dfrac{6me'}{p} \sin m\pi \cos(\bigcirc - \zeta)$,

where $\bigcirc = m\theta + \beta + m\pi$ = sun's longitude at the beginning of the month + $m\pi$

= sun's longitude at the middle of the month.

Hence T will be longest when $\bigcirc - \zeta = 0$,

and shortest when $\bigcirc - \zeta = \pi$;

or T will be longest when the sun at the middle of the month is at perigee, and shortest when at apogee; but, at present, the sun is at perigee about the 1st of January, and apogee about the 1st of July; therefore, owing to *annual equation*, the winter months will be longer than the summer months, the difference between a sidereal month in January and July, from this cause, being about 20 minutes

90 All the inequalities or equations, which our expressions contain, have thus received a physical interpretation They were the only ones known before Newton had established his theory, but the necessity for such corrections was fully recognized, and the values of the coefficients had already been pretty accurately determined; still, with the exception of the reduction, which is geometrically necessary, they were corrections empirically made, and it was scarcely to be expected that any but the larger inequalities, viz those of the first and second orders which we have here discussed, could be detected by observation we find, however, that three others have, since Newton's time, been indicated by observation before theory had explained their cause These are—the *secular acceleration*, discovered by Halley, an inequality, found by Mayer, in the longitude of the moon, and of which the longitude of the ascending node is the argument, and finally an inequality discovered by Burg, which has only within the last five years obtained a solution For a further account of these, as also of some other inequalities which theory has made known, see Appendix, Arts (99), (100), (101), (102)

CHAPTER VII.

APPENDIX.

In this chapter will be found collected a few propositions intimately connected with the results or the processes of the Lunar Theory as explained in the previous pages. Reference has been made to some of them in the course of the work, and the interest and importance of the others are sufficient to justify their introduction here.

91. *The moon is retained in her orbit by the force of gravity, that is, by the same force which acts on bodies at the surface of the earth.*

The proof of this is merely a numerical verification; the data required from observation are,

the space fallen through from rest in 1″ by bodies at the earth's surface = 16·1 feet,

the radius of the earth .. .. .. = 4000 miles,

the periodic time of the moon . . = $27\frac{1}{3}$ days,

the distance of the moon from the earth's centre = 60 × 4000 miles.

The force of the earth's attraction $\propto \frac{1}{(\text{dist.})^2}$. Therefore, the space fallen through in 1″ at distance of moon by a body moving from rest under the earth's action $= \frac{16 \cdot 1}{60^2}$ feet

$= \cdot 00447$ feet

But the moon in one second describes an angle $\frac{2\pi}{27\frac{1}{3}.24.60^2} = \omega$, during which the approach to the earth

$$= 60 \times 4000 \times 5280 \text{ (vers } \omega) \text{ feet}$$

$$= \frac{60 \times 4000 \times 5280 \; 2\pi^2}{(27\frac{1}{3})^2.(24)^2 \; (60)^4} \text{ feet}$$

$$= \cdot 00448 \text{ feet}$$

Therefore, the space through which the moon is deflected in one second from her straight path, is just the quantity through which she would fall towards the earth, supposing her to be subject to the earth's attraction, and we may, therefore, conclude that she is retained in her orbit by the force of gravity

When first Newton, in 1666, attempted to verify this result, he found a difference between the two values equal to one-sixth of the less . the reason of his failure was the incorrect measures of the earth, which he made use of in his computation ; and it was not till about 16 years later that he was led to the true result, by using the more correct value of the earth's radius obtained by Picart *Principia*, lib III , prop 4

92 *The moon's orbit is everywhere concave to the sun*

Let S, E, and M (fig 11) be the centres of the sun, earth, and moon We must bring the sun to rest by applying to the three bodies forces equal and opposite to those which act on the sun ; but these are so small that we may neglect them and consider the moon as moving round the sun fixed, and disturbed by the earth alone

The forces on M are, therefore, $\frac{m'}{SM^2}$ .. . . in MS,

and $\frac{E}{EM^2}$ . . in ME

This last must be resolved into two, one in MS, the other perpendicular to it

Therefore, the whole central force on the moon in

$$MS = \frac{m'}{SM^2} + \frac{E}{EM^2} \cos M,$$

and the proposition will be proved if we shew that this force is always positive

Now, period round sun $= \dfrac{2\pi\, SG^{\frac{3}{2}}}{m'^{\frac{1}{2}}} = \dfrac{2\pi\, SM^{\frac{3}{2}}}{m'^{\frac{1}{2}}}$ nearly,

and earth $= \qquad = \dfrac{2\pi\, EM^{\frac{3}{2}}}{(E+M)^{\frac{1}{2}}}$,

therefore $\dfrac{m'}{SM^3} = \frac{1}{169}\,\dfrac{E+M}{EM^3}$ nearly,

therefore $\dfrac{m'}{SM^3} > \frac{1}{169}\,\dfrac{E}{EM^3}$,

$$\frac{m'}{SM^2} > \frac{1}{169}\,\frac{SM}{EM}\,\frac{E}{EM^2} > \frac{400}{169}\,\frac{E}{EM^2},$$

therefore $\dfrac{m'}{SM^2} - \dfrac{E}{EM^2}$ is positive:

but the least value of the central force corresponds to $\cos M = -1$, and is then $\dfrac{m'}{SM^2} - \dfrac{E}{EM^2}$. It is, therefore, always positive, or the path always concave to the sun.

At new moon the force with which the moon tends to the sun is, therefore, greater than that with which she tends to the earth: the earth being itself in motion in the same direction, and, at that instant, with greater velocity, will easily explain how, notwithstanding this, the moon still revolves about it.

Central and Tangential Disturbing Forces

93. We have hitherto considered the effects of the central and tangential disturbing forces in combination; but it will be interesting to determine to which of them the several inequalities principally owe their existence.

(1) *To determine the effect of the central disturbing force.*

Make $T = 0$;

therefore $$\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0,$$

and this being substituted in the differential equation (β') of Art (30), gives

$$\frac{d^2u}{d\theta^2} + u = \quad .. \quad . \qquad + A\cos(\theta-\alpha) + \; .. \quad ,$$

the solution of which is

$$u = a\{1 + e\cos(\theta-\alpha)\} + \quad \dots \; + \tfrac{1}{2}A\theta\sin(\theta-\alpha).$$

Our first approximate value $u = a\{1 + e\cos(\theta-\alpha)\}$ is thus corrected by a term which, on account of the factor θ, admits of indefinite increase, and thus becomes ultimately a more important term than that with which we started as being very nearly the true value, and which is confirmed as such by observation (22): for the moon's distance, as determined by her parallax, is never much less than 60 times the earth's radius; whereas this new value of u, when θ is very great, would make the distance indefinitely small, and, on the same principle, we see that any solution, which comprises a term of the form $A\theta\sin(\theta-\alpha)$, cannot be an approximate solution except for a small range of values of θ.

Such terms 'if they really had an existence in our system, must end in its destruction, or at least in the total subversion of its present state, but when they do occur, they have their origin, not in the nature of the differential equations, but in the imperfection of our analysis, and in the inadequate representation of the perturbations, and are to be got rid of, or rather included in more general expressions of a periodical nature, by a more refined investigation than that which led us to them The nature of this difficulty will be easily understood from the following reasoning Suppose that a term, such as $a\sin(A\theta + B)$, should exist in the value of u, in which A being extremely minute, the period of the inequality denoted by it would be of great length, then, whatever might be the value of the coefficient a, the inequality would still be always confined within certain limits, and after many ages would return to its former state

'Suppose now that our peculiar mode of arriving at the value of u, led us to this term, not in its real analytical form $a\sin(A\theta + B)$, but by the

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is the equation for determining the constant ε, and in the second approximation, ε would be found from

$$0 = \varepsilon + 2e\sin(\varepsilon-\alpha) + \tfrac{5}{4}e^2\sin 2(\varepsilon-\alpha) + \qquad ,$$

giving different values of ε at each successive approximation 'way of its development in powers of θ, $\alpha + \beta\theta + \gamma\theta^2 +$ &c, and that, not 'at once, but piecemeal, as it were, a first approximation giving us only 'the term α, a second adding the term $\beta\theta$, and so on If we stopped here, 'it is obvious that we should mistake the nature of this inequality, and that 'a really periodical function from the effect of an imperfect approximation, 'would appear under the form of one not periodical These terms 'in the value of u, when they occur, are not superfluous, they are essential 'to its expression, but they lead us to erroneous conclusions as to the 'stability of our system and the general laws of its perturbations, unless 'we keep in view that *they are only parts of series*, the principal parts, 'it is true, when we confine ourselves to intervals of moderate length, 'but which cease to be so after the lapse of very long times, the rest of 'the series acquiring ultimately the preponderance, and compensating the 'want of periodicity of its first terms'—SIR JOHN HERSCHEL, *Encyclopædia Metropolitana*—PHYSICAL ASTRONOMY, p 679

36 To extricate ourselves from this difficulty, and to alter the solution so that none but periodical terms may be introduced, let us again observe that the equation $\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} = a$ of Art (30), which gave the solution U_1 and thus led to the difficulty, is only an approximate form of the first order of the exact equation (β') of the same article. Any value of u, therefore, which satisfies the approximate equation $\frac{d^2u}{d\theta^2} + u = a$ to the first order, and which evades the difficulty mentioned above, may be taken as a solution to the same order of the exact equation (β').

Such a value will be

$$u = a\{1 + e\cos(c\theta - \alpha)\}. \quad \ldots\ldots\ldots\ \ldots. U_1',$$

provided $1 - c^2$ be of the first order at least, for then

$$\frac{d^2u}{d\theta^2} + u = a + ae(1 - c^2)\cos(c\theta - \alpha),$$
$$= a \text{ to the first order}$$

37. The observations hitherto made to check our approximations were extremely rough (22), and carried on only for a short interval; but when they are made with a little more

accuracy, and extended over several revolutions of the moon, it is found that her apse and the plane of her orbit are in constant motion

The above form of the value of u is suggested by our previous knowledge of this motion of the apse, which, as we shall see Art (66), is connected with the value of c here introduced; and there is no doubt that Clairaut, to whom this artifice is due, was led to it by that consideration, and by his acquaintance with the results of Newton's ninth section, which, when translated into analytical language, lead at once to this form of the value of u *

We might, therefore, taking for granted the results of observation, have commenced our approximations at this step, and have at once written down

$$u = a\,\{1 + e\cos(c\theta - \alpha)\},$$

but we should, in so doing, have merely postponed the difficulty to the next step, since there again, as we shall find, the differential equation is of the form

$$\frac{d^2u}{d\theta^2} + u = \text{a function of } \theta,$$

the correct integral of which would be,

$$u = A\cos(\theta - B) + \qquad ,$$

and this would at the next operation bring in a term with θ for a coefficient, which we *now* know must not be We shall, there-

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* Newton has there shewn, that if the angular velocity of the orbit be to that of the body as $G - F$ to G, the additional centripetal force is $\frac{G^2 - F^4}{G^2} h^2u^3$, the original force being μu^2 Therefore

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} + \frac{G^2 - F^2}{G^2}\,u,$$

$$\frac{d^2u}{d\theta^2} + \frac{F^2}{G^2}\,u = \frac{\mu}{h^2} = \frac{F^2}{G^2}\,\frac{\mu}{h_{\prime}^2} = \frac{F^2}{G^2}\,a,$$

$$u = a\left\{1 + e\cos\left(\frac{F}{G}\theta - a\right)\right\},$$

where $\frac{F}{G}$ is the same as our c